Tutorial 1 Partial Derivative and Uncontraint Optimisation

1. $f(x,y) = 5x^3 + 6x^2y^2 + 3xy^3 - 2y + 1$ Find $f(0,1)$, $f(1,0)$, $f(a,2a)$

[Ans: -1, 6, $f(a,2a) = 48a^4 + 5a^3 - 4a + 1$]

2. Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^3 f}{\partial x^2 \partial y}$ of the following equations

2.1. $f(x,y) = 5x^3 + 6x^2y^2 + 3xy^3 - 2y + 1$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 24y$]

2.2. $f(x,y) = 2(x + y^2)^3 + 5xy + 4$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 24y$]

2.3. $f(x,y) = \frac{5}{x-y} + \frac{2}{y} + xy^3$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{30}{(x-y)^4}$]

2.4. $f(x,y) = (2x + 3y)\, e^{4x + 5y}$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 16e^{4x+5y}(8 + 10x + 15y)$]

2.5. $f(x,y) = \ln(5xy) - e^{2x} + 1/y$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 0$]

3. Find $\frac{\partial z}{\partial x}$ where $z = f(x,y)$ from

3.1. $xz^2 = 5x^2z + 3xy - 4$ [Ans: $\frac{\partial z}{\partial x} = \frac{10xz + 3y - z^2}{2xz - 5x^2}$]

3.2. $2xy = 4e^{3xz} + 2yz$ [Ans: $\frac{\partial z}{\partial x} = \frac{y - 6ze^{3xz}}{y + 6xe^{3xz}}$]

4.

4.1 $z = f(x,y) = xy^2 + 3x^2y - 4y + 1$, $x = g(r,s) = r - s^3 + 5$, $y = h(r,s) = 2rs$. Find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial s}$, given $r = 1$, $s = 1$ [Ans: 246, -10]

4.2 $z = f(x,y) = 2x + x^2y^2 - \frac{1}{xy}$, $x = g(u,v,w) = 2u - vw$, $y = h(u,v) = -uv^2$. Find $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w}$.

[Ans: $\frac{\partial z}{\partial u} = 2\left(2 + 2xy^2 + \frac{1}{x^2y}\right) - v^2\left(2x^2y + \frac{1}{xy^2}\right), \frac{\partial z}{\partial v} = -w\left(2 + 2xy^2 + \frac{1}{x^2y}\right) - 2uv\left(2yx^2 + \frac{1}{xy^2}\right), \frac{\partial z}{\partial w} = -v\left(2 + 2y^2x + \frac{1}{x^2y}\right)$]

5. q_A and q_B are demand functions for products A and B, respectively. Determine whether *A* and *B* are competitive, complementary or neither in the following questions:

5.1. $q_A = 5p_A - 3p_B + 50$, $q_B = 6p_B^2 - 2p_A + 15$ [Ans: Complementary]

5.2. $q_A = \dfrac{2}{p_A p_B^2}, q_B = \dfrac{40}{3e^{2p_A}\sqrt{p_B}}$ [Ans: Complementary]

6.

6.1. The demand function for a product is $q = \dfrac{100}{p^{2.2}}$ Find the elasticity of demand for this product. If the original price of 10 goes up by 2%, how much will the demand change?

[Ans: -4.4%]

6.2. Find the partial elasticity of demand for A with respect to p_A, p_B given

$q_A = 20p_A^{-1.5}p_B^{-0.5}$ $p_A = 4, p_B = 1$ [Ans: -1.5, -0.5]

7. Find all the critical points and classify whether they are relative maximum, relative minimum, or saddle point.

7.1. $f(x,y) = -x^2 + 4x - 4y^2 + 8y + 1$ [Ans : max at (2,1,9)]

7.2. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$ [Ans: (-1,2) (-1,-2) saddle, (0,0) min, (-5/3,0) max]

8. A firm seeks to maximize profits. Using the inputs *K* and *L* (capital and labor), it produces based on the production function

$$P = 100K^{0.25}L^{0.25} - 2K - 2L$$

Find the values of *K* and *L* which give the firm maximum production. How much is the maximum profit? [Ans: 156.25,156.25,625]

9. Find all relative maximum and minimum points of the functions:

(i) $Y = 6 - X^2 - Z^2$

(ii) $Y = (X-2)^2 + (Z-4)^4$

(iii) $Y = (X-2)^2 - (Z-4)^4$

(iv) $Y = (X-1)^2 + (Z-1)^2 + (X-1)^2(Z-1)^2$

[Ans: (i) x=0, z=0 maximum, (ii) x=2, z=4 neither, (iii) x=2, z=4, neither, (iv) x=1, z=1, minimum]

10. A firm can sell its product in two countries, A and B, where demand in country A is given by P_A=100-2Q_A and in country B is given by P_B=100-Q_B. Its total output is Q_A+Q_B

which it can produce at a cost of $50(Q_A + Q_B) + \frac{1}{2}(Q_A + Q_B)^2$. Show that its total profit is given by:

$$100Q_A - 2Q_A^{\,2} + 100Q_B - Q_B^2 - 50(Q_A + Q_B) - \tfrac{1}{2}(Q_A + Q_B)^2$$

How much will it sell in the two countries assuming it maximises profits?

[Ans: $Q_A = 7\frac{1}{7}$ $\quad Q_B = 14\frac{2}{7}$]

11. If a firm spends b per unit of output on advertising then it can sell q units of output at a price given by

$$p = 100 - 2q + 8b^{1/2}$$

In order to produce q units of output it costs $4q+q^2$. Show that its profit π is given by

$$\pi = 100q - 2q^2 + 8qb^{1/2} - 4q - q^2 - bq\ .$$

If the firm maximises profits, how much will it spend on advertising and how much will it produce?

[Ans: $q = 18\frac{2}{3}$ $\quad b = 16$]

Additional:

12. In each case, find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$

(a) $f(x, y) = x^3 y + x^2 y^2 + x + y^2$

(b) $f(x, y) = \dfrac{xy}{x^2 + y^2}$

(c) $f(x, y) = x^3 e^{y^2}$

(d) $f(x, y) = x^5 \ln y$

(e) $f(x, y) = \dfrac{x - y}{x + y}$

(f) $f(x, y) = xy^2 - e^{xy}$

13. Use chain rule to find $\dfrac{\partial z}{\partial t}$ when

(a) $z = f(x, y) = x \ln y + y \ln x \qquad , x = t + 1, y = \ln t$

(b) $z = f(x, y) = x^2 + 2y^2 \qquad , x = t - s^2, y = ts$

14. In each case, identify critical points and classify them

(a) $f(x, y) = e^{x+y} + e^{x-y} - \dfrac{3}{2}x - \dfrac{1}{2}y$ [Ans. (-0.3465,0.3465) minimum]

(b) $f(x, y) = \sqrt{1 - x^2 - y^2} + x^2 - 2y^2$ [Ans.(0,0),($\sqrt{3}$/2,0),(- $\sqrt{3}$/2,0)]