

## Assignment 2 EE320 (Section Aj. Kittichai)

Due on Nov., 8<sup>th</sup> 2020 (Sunday)

### Instruction

- 1) Attempt all
- 2) To submit your homework, write your filename as follow **hw2\_Group\_0x**. One point will be deducted if you don't follow the format of suggested filename.

### Question 1:

Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

- 1.1 What type of constant return to scale does the production function exhibit?
- 1.2 Is the production function increasing with respect to K and L?
- 1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.
- 1.4 Use the Hessian matrix. Proof that the production function is concave.

**Question 2:** Define  $f(x,y)$  for all  $(x,y)$  by

$$f(x, y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

- 2.1 Derive the Hessian matrix of  $f(x, y)$ .
- 2.2 Show that  $f(x,y)$  is monotonically convex function. That's, the function is convex for the entire domain sets defining  $f(x,y)$ .
- 2.3 Find the global extrema of  $f(x,y)$ . What type of extrema is it?

### Question 3:

A monopolist faces the market demand given by  $P = Q^{-c}$  where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by  $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$  where "K" and "L" are the level of capital used and

the number of labor employed, respectively. Assume that the unit price of K and L are set equal to “r” and “w”, respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

From now on, assume that  $c = \frac{1}{4}$ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs. Derive the demand for factor inputs, capital and labor.

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

3.5) Confirm your answer with the second-order condition.

Q.1

$$1.1 \quad K_0, L_0 ; \quad Q_0 = 18(0.2 K_0^{-0.4} + 0.8 L_0^{-0.4})^{-2.5}$$

$$tK_0, tL_0 ; \quad Q = 18(0.2)(tK_0)^{-0.4} + 0.8(tL_0)^{-0.4})^{-2.5}$$

$$Q = (t^{-0.4})^{-2.5} 18 [0.2 K_0^{-0.4} + 0.8 L_0^{-0.4}]^{-2.5}$$

$$Q = t' \cdot Q_0$$

$\therefore$  Degree of H.M. = 1

f is constant return to scale.

Question 1:

Given the equation for the production function

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- 1.1 What type of constant return to scale does the production function exhibit?
- 1.2 Is the production function increasing with respect to K and L?
- 1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.
- 1.4 Use the Hessian matrix. Proof that the production function is concave.

$$1.2) Q = 18 \cdot (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$$

$$\frac{\partial Q}{\partial K} = 18 \cdot \frac{d}{dK} (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$$

$$= 18 \left[ -2.5 (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.08K^{-1.4}) \right]$$

$$\therefore 18 \left[ -2.5 (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.08K^{-1.4}) \right] > 0, \forall K \text{ and } \forall L$$

(K ↑, Q ↑)

$$\frac{\partial Q}{\partial L} = 18 \cdot \frac{d}{dL} (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$$

$$= 18 \left[ -2.5 (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.32L^{-1.4}) \right]$$

$$\therefore \left[ -2.5 (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.32L^{-1.4}) \right] > 0, \forall K \text{ and } \forall L$$

(L ↑, Q ↑)

Ans

$$\text{Total differentiation of } Q = DQ = \frac{\partial Q}{\partial L} \cdot dL + \frac{\partial Q}{\partial K} \cdot dK$$

with respect to all variables

$$= \left[ -2.5 (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.32L^{-1.4}) \right] \cdot dL + 18 \left[ -2.5 (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.08K^{-1.4}) \right] \cdot dK$$

1.3) implicit function :  $\frac{dy}{dx} = - \frac{F_x}{F_y}$

MRTS of L for K =  $\frac{dL}{dK} = - \frac{F_K}{F_L}$

$$Q = 18 \cdot (0.2k^{-0.4} + 0.8L^{-0.4})^{-2.5}$$

$$0 = 18 \cdot (0.2k^{-0.4} + 0.8L^{-0.4})^{-2.5} - Q$$

$$f(Q, k, L) = 18 \cdot (0.2k^{-0.4} + 0.8L^{-0.4})^{-2.5} - Q$$

$$F_L = 18 \left[ -2.5(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.32L^{-1.4}) \right]$$

$$F_K = 18 \left[ -2.5(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.08k^{-1.4}) \right]$$

$$\frac{dL}{dK} = - \frac{F_K}{F_L} = - \frac{18 \left[ -2.5(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.08k^{-1.4}) \right]}{18 \left[ -2.5(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.32L^{-1.4}) \right]}$$

$$0 = 18 \cdot (0.2k^{-0.4} + 0.8L^{-0.4})^{-2.5} - Q$$

$$0 = 18 \left( -2.5(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.08k^{-1.4} + (-0.32L^{-1.4}) \frac{dL}{dK} \right)$$

$$0 = (18)(-2.5)(0.2k^{-0.4} + 0.8L^{-0.4})^{-3.5} (-0.08k^{-1.4} - 0.32L^{-1.4} \frac{dL}{dK})$$

$$0 = -45(-0.08k^{-1.4} - 0.32L^{-1.4} \frac{dL}{dK})$$

$$0 = 3.6k^{-1.4} + 9L^{-1.4} \frac{dL}{dK}$$

$$-3.6k^{-1.4} = 9L^{-1.4} \frac{dL}{dK}$$

$$\frac{-3.6k^{-1.4}}{9L^{-1.4}} = \frac{dL}{dK}$$

$$\frac{-3.6k^{-1.4}}{9L^{-1.4}} = \text{MRTS of L with respect to K}$$

< 0 for  $\forall L, \forall K$

1.4 Use the Hessian matrix. Proof that the production function is concave.

① Check the properties of H matrix

•  $Q = 18 \cdot (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$

•  $f(K,L) = 18 \cdot (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$

•  $f_K = 18 \left[ -2.5(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.08K^{-1.4}) \right]$      •  $f_L = 18 \left[ -2.5(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5} \cdot (-0.32L^{-1.4}) \right]$

$\swarrow$  ~~FKK~~  $\downarrow$

$= 3.6 \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right)^{-3.5} \cdot (K^{-1.4})$

$= 3.6 \left[ \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right)^{-3.5} \cdot (-1.4K^{-2.4}) + \left( K^{-1.4} \cdot \left[ (-3.5) \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right)^{-4.5} \cdot (-0.08K^{-1.4}) \right] \right) \right]$

(A) = < 0

$\downarrow$  ~~FLK~~  $\swarrow$

$3.6 \left( K^{-1.4} \cdot \left( -3.5 \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right)^{-4.5} \cdot (-0.32L^{-1.4}) \right) \right)$

$= (-0.32)(3.6)(K^{-1.4})(-3.5)(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5}(L^{-1.4})$

$= 4.032(K^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})(L^{-1.4})$

(B) = > 0

$\swarrow$  ~~FLF~~  $\downarrow$

$= 14.4 \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right)^{-3.5} (L^{-1.4})$

$= 14.4 \left[ \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right) (-1.4L^{-2.4}) + (L^{-1.4})(-3.5)(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5} (-3.2L^{-1.4}) \right]$

$= 14.4 \left[ \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right) (-14L^{-2.4}) + (11.2)(L^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5} \right]$

(C) = < 0  $\forall K, \forall L$

$\downarrow$

$= 14.4 \left( L^{-1.4} (-3.5) \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right)^{-3.5} (-0.08K^{-1.4}) \right)$

$= 4.032 \left( L^{-1.4} \right) \left( 0.2K^{-0.4} + 0.8L^{-0.4} \right)^{-3.5} (K^{-1.4})$

(D) = > 0 ,  $\forall K, \forall L$



Question 2: Define  $f(x,y)$  for all  $(x,y)$  by

$$f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

2.1 Derive the Hessian matrix of  $f(x,y)$ .

2.2 Show that  $f(x,y)$  is monotonically convex function. That's, the function is convex for the entire domain sets defining  $f(x,y)$ .

2.3 Find the global extrema of  $f(x,y)$ . What type of extrema is it?

$$f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

$$f_x = e^{x+y} + e^{x-y} - \frac{3}{2}$$

$$f_y = e^{x+y} + e^{x-y}(-1) - \frac{1}{2}$$

$$= e^{x+y} - e^{x-y} - \frac{1}{2}$$

$$f_{xx} = e^{x+y} + e^{x-y}$$

$$f_{xy} = e^{x+y} + e^{x-y}(-1)$$

$$= e^{x+y} - e^{x-y}$$

$$f_{yy} = e^{x+y} + e^{x-y}$$

$$f_{yx} = e^{x+y} - e^{x-y}$$

2.1). Derive the Hessian matrix of  $f(x,y)$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$$

$$[H_1] = > 0, \forall x, \forall y$$

$$[H_2] = > 0, \forall x, \forall y$$

(Ans)

$\therefore H$  is positive definite,  $d^2f > 0$  for all value of  $x, y$

(2.2)  $\rightarrow$  globally convex  $\rightarrow$  global extrema (Ans)

$$(2.3) \quad f_x = e^{x+y} + e^{x-y} - \frac{3}{2}$$

$$f_y = e^{x+y} - e^{x-y} - \frac{1}{2}$$

$$0 = e^{x+y} + e^{x-y} - \frac{3}{2}$$

$$\frac{3}{2} = e^{x+y} + e^{x-y}$$

$$\ln \frac{3}{2} = \ln e^{x+y} + \ln e^{x-y}$$

$$\ln \frac{3}{2} = x+y + (x-y)$$

$$\ln \frac{3}{2} = 2x$$

$$0.405 = 2x$$

$$0.2 = x$$

$$\frac{1}{2} = e^{x+y} - e^{x-y}$$

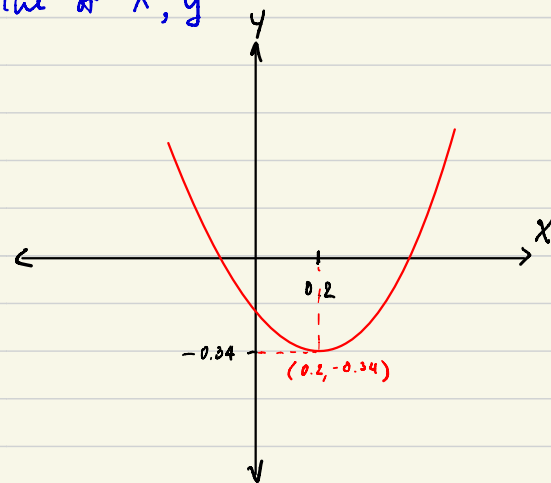
$$\ln \frac{1}{2} = \ln e^{x+y} - \ln e^{x-y}$$

$$\ln \frac{1}{2} = x+y - x+y$$

$$\ln \frac{1}{2} = 2y$$

$$-0.34 = 2y$$

$$(0.2, -0.34)$$



global min

(Ans)

$\therefore f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$  is globally convex at point  $(0.2, -0.34)$

3.1) What type of the return to scale technology does the production function exhibit?

$$Q = K^{\frac{1}{3}} L^{\frac{2}{3}}$$

$$\begin{aligned} Q' &= (+K)^{\frac{1}{3}} (+L)^{\frac{2}{3}} \\ &= +\frac{1}{3} K^{\frac{1}{3}} \cdot +\frac{2}{3} L^{\frac{2}{3}} \\ &= +\frac{2}{3} \cdot K^{\frac{1}{3}} \cdot L^{\frac{2}{3}} \end{aligned}$$

$\therefore$  degree of Homogeneity = 1  $Q(K,L) \Rightarrow$  constant return to scale

From now on, assume that  $c = \frac{1}{4}$ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$\begin{aligned} \pi &= (P \cdot Q) - \text{cost} \\ &= \left( Q^{-\frac{1}{4}} \cdot K^{\frac{1}{3}} L^{\frac{2}{3}} \right) - (wL + rK) \\ &= \left( K^{\frac{1}{3}} L^{\frac{2}{3}} \right)^{-\frac{1}{4}} \cdot K^{\frac{1}{3}} L^{\frac{2}{3}} - wL - rK \\ &= K^{\frac{3}{12}} L^{\frac{6}{12}} - wL + rK \end{aligned}$$

$$\therefore \pi = K^{\frac{1}{4}} L^{\frac{1}{2}} - wL + rK$$

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs. Derive the demand for factor inputs, capital and labor.

$$\frac{d\pi}{dK} = \frac{1}{4} \cdot L^{\frac{1}{2}} K^{-\frac{3}{4}} - r = 0$$

$$\begin{aligned} K^{-\frac{3}{4}} &= \frac{4r}{\sqrt{L}} \\ K &= \left( \frac{4r}{\sqrt{L}} \right)^{-\frac{4}{3}} \end{aligned}$$

$$L \rightarrow K : K^* = \left[ \left( \frac{\left( \frac{4\sqrt{w}}{2w} \right)^2}{4r} \right)^{\frac{1}{2}} \right]^{\frac{1}{3}}$$

$$= \left( \frac{1}{8rw} \right)^{\frac{8}{9}}$$

$$\frac{d\pi}{dL} = \frac{1}{2} \cdot K^{\frac{1}{4}} L^{-\frac{1}{2}} - w = 0$$

$$L^{-\frac{1}{2}} = \frac{2w}{\sqrt[4]{K}}$$

$$L = \left( \frac{2w}{\sqrt[4]{K}} \right)^{-2}$$

$$K^* \rightarrow L^* = \left[ \left( \frac{1}{8rw} \right)^{\frac{8}{9}} \right]^{\frac{1}{4}} \cdot \frac{1}{2w}$$

$$= \frac{1}{4w} \left( \frac{1}{8rw} \right)^{\frac{1}{9}}$$

3.4) How does the demand for labor vary with respect to  $w$  and  $r$ ? Show your result by using partial derivative.

$$\begin{aligned} \frac{\partial L^*}{\partial w} &: \frac{1}{4w} \cdot (8r)^{-\frac{4}{9}} \cdot w^{-\frac{13}{9}} \cdot \frac{-4}{9} + \left( \frac{1}{8rw} \right)^{\frac{4}{9}} \cdot \frac{1}{4} \cdot -1 \cdot w^{-2} \\ &= \frac{-1}{9((8r)^{\frac{4}{9}} \cdot w^{\frac{22}{9}})} + \frac{-1}{4((8r)^{\frac{4}{9}} \cdot w^{\frac{22}{9}})} \\ &= \frac{-13}{36((8r)^{\frac{4}{9}} \cdot w^{\frac{22}{9}})} \end{aligned}$$

$$\begin{aligned} \frac{\partial L^*}{\partial r} &: \frac{1}{4w} (8rw)^{-\frac{4}{9}} \\ &= \frac{1}{4w} (8w)^{-\frac{4}{9}} \cdot (r)^{-\frac{4}{9}} \\ &= \frac{1}{4w} (8w)^{-\frac{4}{9}} \cdot \left(-\frac{4}{9}\right) \cdot r^{-\frac{13}{9}} \\ &= \frac{-1}{9(18w)^{\frac{4}{9}} \cdot r^{\frac{13}{9}}} \end{aligned}$$

3.5) Confirm your answer with the second-order condition.

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \cdot L^{\frac{1}{2}} \cdot \left(-\frac{3}{4}\right) \cdot K^{-\frac{1}{4}} & \frac{1}{4} \cdot K^{-\frac{3}{4}} \cdot \frac{1}{2} \cdot L^{-\frac{1}{2}} \\ \frac{1}{2} \cdot \frac{1}{4} \cdot L^{-\frac{1}{2}} \cdot K^{-\frac{3}{4}} & \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot K^{\frac{1}{4}} \cdot L^{-\frac{3}{2}} \end{bmatrix}$$

$$|H_1| = \frac{1}{4} \cdot L^{\frac{1}{2}} \cdot \left(-\frac{3}{4}\right) \cdot K^{-\frac{1}{4}} = \frac{-3\sqrt{L}}{16K^{\frac{1}{4}}\sqrt{K^3}}$$

$$\begin{aligned} |H_2| &= \left( \frac{1}{4} \cdot L^{\frac{1}{2}} \cdot \left(-\frac{3}{4}\right) \cdot K^{-\frac{1}{4}} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot K^{\frac{1}{4}} \cdot L^{-\frac{3}{2}} \right) - \left( \frac{1}{2} \cdot \frac{1}{4} \cdot L^{-\frac{1}{2}} \cdot K^{-\frac{3}{4}} \cdot \frac{1}{4} \cdot K^{-\frac{3}{4}} \cdot \frac{1}{2} \cdot L^{-\frac{1}{2}} \right) \\ &= \frac{-7}{64LK\sqrt{L}} \end{aligned}$$

$$\text{plug } K = \left(\frac{4r}{\sqrt{L}}\right)^{\frac{4}{3}}, \quad L = \left(\frac{2w}{4\sqrt{K}}\right)^{-2} \quad \text{in } |H_1|, |H_2|$$

$$|H_1| = \frac{-3 \left(\frac{4\sqrt{K}}{2w}\right)}{16 \cdot \left(\frac{\sqrt{L}}{4r}\right)^{\frac{4}{3}}} < 0$$

$\therefore$  At  $K, L \Rightarrow H$  is definite;  $d^2\pi < 0$   
So  $\pi$  is globally concave

$$|H_2| = \frac{-7}{64 \left(\frac{\sqrt{L}}{4r}\right)^{\frac{4}{3}} \cdot \left(\frac{4\sqrt{K}}{2w}\right)^3} < 0$$