Exercise 3 (Part 1)

1. (a) Prove the statement:
   "There is a pair of real numbers x and y such that $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. "

   (b) Disprove the statement: "For all real numbers x and y, $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. "

2. Show that "for any integer n, if $n^3 + 5$ is odd, then n is even," by using
   a) a proof by contraposition,
   b) a proof by contradiction.

3. Prove by the **method of exhaustion** that " $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$."

4. Use the **proof by cases** to show that " for any integer n, $n^2 \geq n$."
   [Hint: Consider 3 cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$]

5. Consider the statement: for $n \geq 1$,
   $$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$
   Suppose we want to prove the above statement by **mathematical induction**.
   (a) What is $P(n)$?
   (b) Write $P(1)$: Is $P(1)$ true?
   (c) Write $P(k)$:
   (d) Write $P(k+1)$:
   (e) Prove the above statement: $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$, by using **mathematical induction**.

6. Use mathematical induction proof to show that
   $$n! < n^n,$$
   for any integer n that is greater than 1.

7. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

8. (Optional) Use the method of constructive proof to show that:
   if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

9. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

10. (Optional) A sequence $a_1, a_2, \ldots$ is defined recursively by
    $$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$
    Show that
    $$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$