Answer to Problem Set 1

The starting point is the value function for t=2. Since x_3=0, the transition equation for period 2 is given as

$$x_3 = 0 = x_2 - u_2 \tag{1}$$

The transition equation for t=2 provides us with the optimal decision rule for the control variable u in period 2.

$$u_2 = x_2 \tag{2}$$

Equation 2 simply says to extract and sell all the oil that remains at the beginning of t=2 in that same period. It is independent of any price variables or any other variables dates t<2. It is an application of the principle of optimality that underlying dynamic programming: we can do no better than just follow the derived optimality policy rule regardless of what happens to control and state variables in earlier periods.

To obtain the optimal decision rules for the control variable in the period prior to t=2, we will make use of the recursive nature of Bellman equation. One period at a time, we will move backward in time until we have found the decision rule for t=0.

The first step in finding the decision rule for setting the control variable in t=1 consists of substituting the optimal policy function for t=2 into the value function for t=2,

$$V_1 = \left(p_2 u_2 - 0.05u_2^2 \right) + 0.9V_0 \left(x_2 - u_2 \right)$$

By equation 2, this is simplified to

$$V_1 = \left(p_2 x_2 - 0.05x_2^2 \right) \tag{3}$$

Next, we substituting equation 3 into the Bell equation for period t=1

$$V_2 = \max_{u_1} \left(p_1 u_1 - 0.05u_1^2 \right) + 0.9V_1$$

To obtain

$$V_2 = \max_{u_1} \left(p_1 u_1 - 0.05u_1^2 \right) + 0.9\left(p_2 x_2 - 0.05x_2^2 \right) \tag{4}$$

Before we can maximize the right-hand side of equation 4 with respect to u_1 we need to consider that x_2 is connected to u_1 via the transition function

$$x_2 = x_1 - u_1 \tag{5}$$

Substituting equation 5 into 4 we obtain

$$V_2 = \max_{u_1} \left(p_1 u_1 - 0.05u_1^2 \right) + 0.9\left(p_2 \left(x_1 - u_1 \right) - 0.05\left(x_1 - u_1 \right)^2 \right) \tag{6}$$

Now, only the decision variable u_1 and variables that are known at time t=1 are in equation 6. The right-hand side of equation 6 can be maximized with respect to the control variable u_1,

$$\frac{d\left[\left(p_1u_1 - 0.05u_1^2\right) + 0.9\left(p_2\left(x_1 - u_1\right) - 0.05\left(x_1 - u_1\right)^2\right)\right]}{du_1} = 0$$

$$p_1 - 0.19u_1 - 0.9p_2 + 0.09u_1 = 0.$$

Solving the above first-order condition for u_1 gives the optimal policy rule for period 1,

$$u_1 = 0.4737x_1 + 5.263p_1 - 4.737p_2 \qquad (7)$$

This complete the calculation for t=1.

To find optimal policy rule for period 0, we need to insert equation 7 into the value function for period 1 to get an expression similar to equation 3. The value function for period 1 can be written as

$$V_2 = 0.474\left(p_1 + p_2\right)x_1 + 2.636p_1^2 - 4.734p_1p_2 - 0.02368x_1^2 + 2.132p_2^2$$

The value function for period 1 needs to be substituted into the Bellman equation for period 0

$$V_3 = \max_{u_0}\left(p_0u_0 - 0.05u_0^2\right) + 0.9V_2$$

One obtains

$$V_3 = \max_{u_0}\left(p_0u_0 - 0.05u_0^2\right) + 0.9\left[0.474\left(p_1 + p_2\right)x_1 + 2.636p_1^2 - 4.734p_1p_2 - 0.02368x_1^2 + 2.132p_2^2\right] \qquad (8)$$

Making use of the transition equation

$$x_1 = x_0 - u_0 \qquad (9)$$

in equation 8, one can maximize its right-hand side with respect to u_0,

$$\frac{d\left[\left(p_0u_0 - 0.05u_0^2\right) + 0.9\left\{0.474\left(p_1 + p_2\right)\left(x_0 - u_0\right) + 2.636p_1^2 - 4.734p_1p_2 - 0.02368\left(x_0 - u_0\right)^2 + 2.132p_2^2\right\}\right]}{du_0} = 0$$

Solving the resulting first-order condition for u_0 one obtains

$$u_0 = 0.2989x_0 + 7.013p_0 - 2.989p_1 - 2.989p_2$$

This is the optimal policy or decision rule for period 0.