Solution: Assignment 3

1. (a) Prove the statement:
"There are distinct integers m and n such that $\frac{1}{m} + \frac{3}{n}$ is an integer. "

(b) Disprove the statement: " For all integers x and y, if $3x+y$ is odd then x and y are both odd. "

Answer:

(a) The statement:
"There are distinct integers m and n such that $\frac{1}{m} + \frac{3}{n}$ is an integer " is an existential statement. So, we can prove this by give some specific values of m and n that make this statement true. Consider $m = 2$ and $n = 6$. Then

$$\frac{1}{m} + \frac{3}{n} = \frac{1}{2} + \frac{3}{6} = 1,$$

which is an integer. Therefore, the given statement is true. ■

(b) The statement " For all integers x and y, if $3x + y$ is odd then x and y are both odd " is a universal statement. We can **disprove** this statement by finding a counterexample that make this statement false or make its **negation** true. The negation is given by

" there exist integers x and y, such that $3x + y$ is odd, and (but) x or y is even "

Let $x = 2$ and $y = 1$. Notice that x is even and $3x + y = 3(2) + 1 = 5$ is odd. Hence, a counterexample is $x = 2$, $y = 1$ and we disprove the given statement. ■

2. Show that "for all integers m and n, if mn is even, then m is even or n is even," by using
(a) a proof by contraposition,
(b) a proof by contradiction.

Answer:

(a)**Proof by contraposition:**

The contrapositive of the given statement is

"for all integers m and n, if m is odd and n is odd, then mn is odd."

Let $m, n \in \mathbb{Z}$. Suppose m and n are odd. Then we can write
$m = 2k + 1$, for some $k \in \mathbb{Z}$ and
$m = 2s + 1$, for some $s \in \mathbb{Z}$
Consider mn:

$$mn = (2k + 1)(2s + 1) = 4ks + 2k + 2s + 1 = 2\underbrace{(2ks + k + s)}_{=:r\in\mathbb{Z}} + 1 = 2r + 1,$$

which is an odd integer, by definition. Note that $r = 2ks + k + s$ is an integer since it is the sum of product of integers $k, s \in \mathbb{Z}$.

The contrapositive is equivalent to the original statement and therefore showing that its contrapositive is true implies that the original statement is also true. ■

(b) **Proof by contradiction:**

Suppose not. I.e. suppose that its negation is true:

"there exist integers m and n, such that mn is even, and m, n are both odd."

Then since m, n are both odd, we can write $m = 2k + 1$, for some $k \in \mathbb{Z}$ and
$m = 2s + 1$, for some $s \in \mathbb{Z}$
Consider mn:

$$mn = (2k+1)(2s+1) = 4ks + 2k + 2s + 1 = 2\underbrace{(2ks + k + s)}_{=:r\in\mathbb{Z}} + 1 = 2r + 1,$$

which is an odd integer, by definition, since $r = 2ks + k + s$ is an integer.

This contradicts to the assumption that mn is even. Hence its negation is false and the statement itself is true. ■

3. Prove by the **method of exhaustion** that " $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$."

   Answer Given that an integer n such that $1 \leq n \leq 4$ implies that $n = 1, 2, 3,$ or 4.

   For $n = 1$, $1^2 + 1 = 2^1$ and $n^2 + 1 \geq 2^n$ is true.
   For $n = 2$, $2^2 + 1 = 5$ and $2^2 = 4$, so $n^2 + 1 \geq 2^n$ is true.
   For $n = 3$, $3^2 + 1 = 10$ and $2^3 = 8$, so $n^2 + 1 \geq 2^n$ is true.
   For $n = 4$, $4^2 + 1 = 17$ and $2^4 = 16$, so $n^2 + 1 \geq 2^n$ is true.

   Therefore, $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$. ■

4. Use the **proof by cases** to show that " Prove that for all integers m and n,
   $m + n$ and $m - n$ are either both odd or both even."

   [Hint: Consider 4 cases of even and odd for m and n]

   Answer: We will consider 4 cases for m and n.

   - Case I: m is even and n is even.
     That is, $m = 2k$ and $n = 2s$ for some integers $k, s \in \mathbb{Z}$. which implies $m - n$ is even because $k - s$ is an integer.
     $$m + n = 2k + 2s = 2(k + s)$$
     Then, which implies $m + n$ is even because $k + s$ is an integer. Also,
     $$m - n = 2k - 2s = 2(k - s)$$
     Hence, both $m + n$ and $m - n$ are both even in this case.

- Case II: m is even and n is odd.
  That is, $m = 2k$ and $n = 2a + 1$ for some integers $k, s \in \mathbb{Z}$. Then,

  $$m + n = 2k + (2a + 1) = 2(k + a) + 1$$

  which implies $m + n$ is odd because $k + a$ is an integer. Also,

  $$m - n = 2k - (2a + 1) = 2(k - a) - 1 = 2(k - a) - 1 + 2 - 2 = 2(k - a - 1) + 1$$

  which implies $m - n$ is odd because $k - a - 1$ is an integer. Hence, both $m + n$ and $m - n$ are both odd in this case.

- Case III: m is odd and n is even. That is, $m = 2b+1$ and $n = 2s$ for some integers $k, s \in \mathbb{Z}$. Then,

  $$m + n = (2b + 1) + 2s = 2(b + s) + 1$$

  which implies $m + n$ is odd because $b + s$ is an integer. Also,

  $$m - n = (2b + 1) - 2s = 2(b - s) + 1$$

  which implies $m - n$ is odd because $b - s$ is an integer. Hence, both $m + n$ and $m - n$ are both odd in this case.

- Case IV: m is odd and n is odd.
  That is, $m = 2b + 1$ and $n = 2a + 1$ for some integers $k, s \in \mathbb{Z}$. Then,

  $$m + n = (2b + 1) + (2a + 1) = 2(b + a + 1)$$

  which implies $m + n$ is even, by definition, because $b + a + 1$ is an integer. Also,

  $$m - n = (2b + 1) - (2a + 1) = 2(b - a)$$

  which implies $m - n$ is odd, by definition, because $b - a$ is an integer. Hence, both $m + n$ and $m - n$ are both even in this case. ■

5. Consider the statement: for any integer $n \geq 0$,

   $$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^n = \frac{1}{2}(5^{n+1} - 1).$$

   Suppose we want to prove the above statement by **mathematical induction**.

   (a) What is $P(n)$?
   (b) Write $P(0)$: Is $P(0)$ true?
   (c) Write $P(k)$:
   (d) Write $P(k + 1)$:
   (e) Prove the above statement: $\sum_{j=0}^{n} 2 \cdot 5^j = \frac{1}{2}(5^{n+1} - 1)$ by using **mathematical induction**.

   Answer:

(a) What is $P(n)$?
$P(n)$ is the statement

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^n = \frac{1}{2}(5^{n+1} - 1).$$

(b) Write $P(0)$: Is $P(0)$ true?
$P(0) : 2 = \frac{1}{2}(5^{0+1} - 1) = 2$ is true.

(c) Write $P(k)$:

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^k = \frac{1}{2}(5^{k+1} - 1).$$

(d) Write $P(k+1)$:

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^{k+1} = \frac{1}{2}(5^{k+2} - 1).$$

(e) Prove the above statement: $\sum_{j=0}^{n} 2 \cdot 5^j = \frac{1}{2}(5^{n+1} - 1)$ by using **mathematical induction**.
We want to prove that $P(n)$ is true for all integer $n \geq 0$.
(I) **Basis step:** $P(0) : 2 = \frac{1}{2}(5^{0+1} - 1) = 2$ is true.
(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 0$.
Assume that $P(k)$ is true:
$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^k = \frac{1}{2}(5^{k+1} - 1)$. ———— (★) "inductive hypothesis"
We want to show that $P(k+1)$ is true:
$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^{k+1} = \frac{1}{2}(5^{k+2} - 1)$.

$$\begin{aligned}
\underbrace{2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^k}_{(\bigstar)} + 2 \cdot 5^{k+1} &= \frac{1}{2}(5^{k+1} - 1) + 2 \cdot 5^{k+1} \text{ by } (\bigstar) \\
&= 5^{k+1} \underbrace{\left(\frac{1}{2} + 2\right)}_{5/2} - \frac{1}{2} \\
&= \frac{5^{k+2}}{2} - \frac{1}{2} \\
&= \frac{1}{2}(5^{k+2} - 1),
\end{aligned}$$

which implies that $P(k+1)$ is true. Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 0$ by the induction proof. ■

6. Use mathematical induction proof to show that

$$2^n > n^2$$

for an integer greater than 4.

Answer:

Let $P(n)$ be the statement $2^n > n^2$.

We want to prove that $P(n)$ is true for all integer $n \geq 5$.

(I) **Basis step:** $P(5) : 2^5 > n^2$ or $32 > 25$, which is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 5$.
Assume that $P(k)$ is true:
$P(k)$: $2^k > k^2$. ———(★) "inductive hypothesis"
We want to show that $P(k+1)$ is true: $2^{k+1} > (k+1)^2$.

Note that for $k \geq 5$, we have $(k-2) \geq 3$ and $k(k-2) \geq 15 > 1$:

$$\begin{aligned} k(k-2) &> 1 \\ k^2 - 2k &> 1 \\ k^2 + k^2 - 2k &> k^2 + 1 \\ 2k^2 &> k^2 + 2k + 1 = (k+1)^2. \end{aligned}$$

That is, $\boxed{2k^2 > (k+1)^2 \text{ for any integer } k \geq 5}$. ————————-(♠)

Consider 2^{k+1} from $P(k+1)$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 && \text{——————by "inductive hypothesis" (★) } 2^k > k^2 \\ &> (k+1)^2 && \text{——————by (♠) } 2k^2 > (k+1)^2 \end{aligned}$$

which implies that $P(k+1)$ is true.

Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 0$ by the induction proof. ■

Remark: To show (♠), it is also possible to use the proof by induction (see next page).

Note that to show (♠), it is also possible to use the proof by induction. Let $Q(n)$ be the statement $\boxed{2n^2 > (n+1)^2}$.

We want to prove that $Q(n)$ is true for all integer $n \geq 5$.
(i) **Basis step:** $Q(5) : 2 \cdot 5^2 > (5+1)^2$ or $50 > 36$, which is true.
(ii) **Inductive step:** Assume that $Q(k)$: $2k^2 > (k+1)^2$——(★★).
We want to show that $Q(k+1)$: $2(k+1)^2 > (k+2)^2$.

$$\begin{aligned}
2(k+1)^2 &= 2k^2 + 4k + 2 \\
&> (k+1)^2 + 4k + 2 \qquad \text{by "inductive hypothesis" (★★) } 2^k > k^2 \\
&= k^2 + 2k + 1 + 4k + 2 \\
&= (k^2 + 4k + 4) + \underbrace{2k - 1} \\
&= (k+2)^2 + \underbrace{2k-1}_{\geq 0} \\
&> (k+2)^2 \qquad \text{since } 2k - 1 > 0 \text{ for } k \geq 5
\end{aligned}$$

which implies that $2(k+1)^2 > (k+2)^2$ and $P(k+1)$ is true.
From (i) and (ii), $\boxed{2n^2 > (n+1)^2}$ for all $n \geq 5$.

7. (Optional) Use the method of constructive proof to show that:
if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

Answer: Constructive proof
Let $r, s \in \mathbb{R}$ such that $r < s$. Let

$$x = \frac{r+s}{2}.$$

We will show that for this particular x has the value between the r and s.

$$\begin{aligned}
r &< s \\
r + r &< s + r \\
\underbrace{\frac{r+r}{2}}_{=r} &< \underbrace{\frac{s+r}{2}}_{=x} \quad \Rightarrow \quad r < x
\end{aligned}$$

$$\begin{aligned}
r &< s \\
r + s &< s + s \\
\underbrace{\frac{r+s}{2}}_{=x} &< \underbrace{\frac{s+s}{2}}_{=s} \quad \Rightarrow \quad x < s
\end{aligned}$$

That is, for any given r and s, we can always find $x = \frac{r+s}{2}$ such that $r < x < s$.

Note that it is also possible to use a different value of x. ■

8. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

   Answer: Let r be any rational number and s be any irrational number. We want to show that $r - s$ is irrational.

   To prove this by contradiction, we will suppose that $r - s$ is rational. Then we can write $r = \frac{a}{b}$ and $r - s = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$. That is,

$$\frac{a}{b} - s = \frac{c}{d}$$

   and so

$$s = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

   which implies that s is rational. This is a contradiction to the assumption that s is irrational. Therefore, the the given statement is true by contradiction proof. ■

9. (Optional) A sequence $a_1, a_2, \ldots$ is defined recursively by

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$

   Show that

$$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$

   Answer:
   Proof by mathematical induction:
   From the given definition :

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2. \qquad ---------------\circledast$$

   Let $P(n)$ be the statement $a_n = 3 \cdot 7^{n-1}$.
   We want to prove that $P(n)$ is true for all integer $n \geq 1$.

   (I) **Basis step:** Show that $P(1)$ is true.
   $P(1)$: $a_1 = 3 \cdot 7^{1-1}$.
   Since $a_1 = 3 \cdot 7^{1-1} = 3 \cdot 7^0 = 3 \cdot 1 = 3$, which is the same as $a_1 = 3$ from the definition $\circledast$.
   Hence $P(1)$ is true.

   (II)**Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.

   Assume that $P(k) : a_k = 3 \cdot 7^{k-1}$ is true. ————————($\bigstar$) "inductive hypothesis"
   We want to show that $P(k+1) : a_{k+1} = 3 \cdot 7^{(k+1)-1}$, or $a_{k+1} = 3 \cdot 7^k$ is true. Consider from the definition $\circledast$

$$\begin{aligned} a_{k+1} &= 7a_{[(k+1)-1]} \\ &= 7\ a_k \\ &= 7\ [3 \cdot 7^{k-1}] \qquad \text{by } (\bigstar) \text{ "inductive hypothesis : } a_k = 3 \cdot 7^{k-1} \text{ "} \\ &= 3 \cdot 7^k \end{aligned}$$

   and therefore $P(k+1)$ is true.

   From (I) basis step and (II) inductive step, $P(n)$ is true for any integer $n \geq 1$ by mathematical induction proof. ■