EE320 (1/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION UNDER EQUALITY CONSTRAINTS

Topics

- Effects of a constraint
- Finding the stationary (optimal) values
  - Elimination method
  - Lagrange multiplier
- Second-order conditions for constrained optimization
- Economic applications
- Extensions: n-variable and multiconstraint cases

Effects of a Constraint

- In the previous topic, we studied optimization problems, where the choice variables can be chosen freely.
- Example: A profit-maximizing firm's objective is:

$$\pi = PQ - [C(Q_1) + C(Q_2)]$$

where $Q = Q_1 + Q_2$.

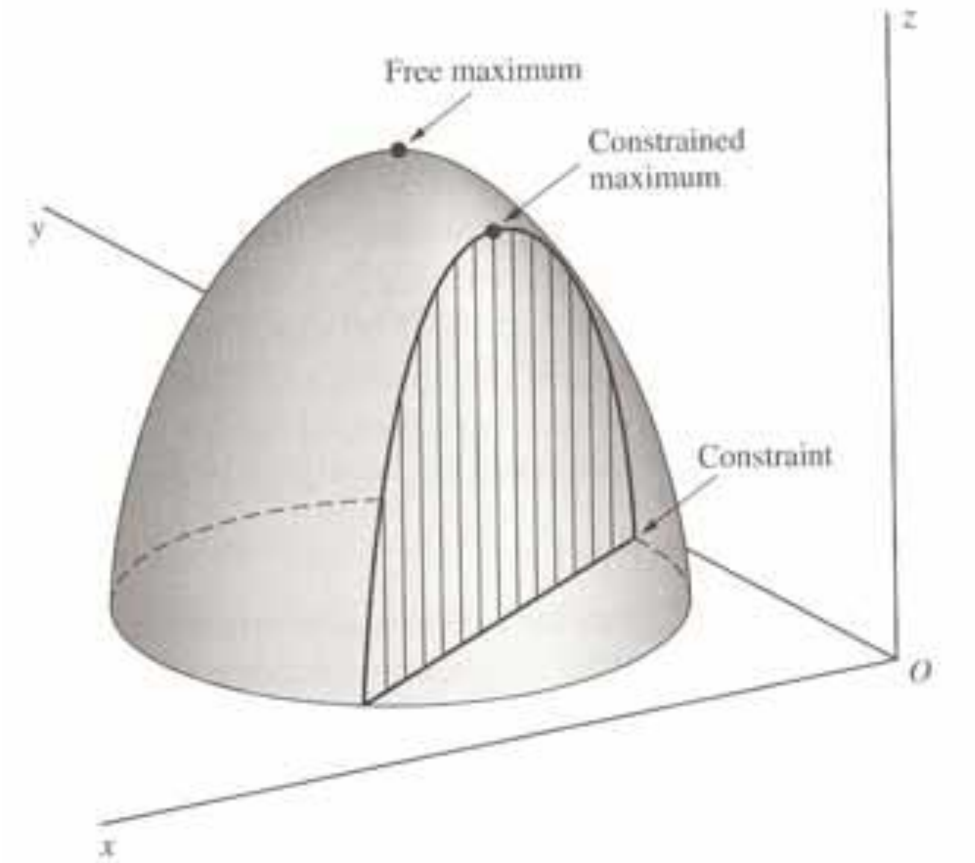
- Q_1 and Q_2 can be chosen freely in order to maximize profit.

  ➔ Here, Q_1^* and Q_2^* are *free optimal values*.

- Question: If there is a *production constraint*, e.g. $Q_1 + Q_2 = 900$, then how would the firm change its behavior?
- ➔ Q_1^{**} and Q_2^{**} will now become the *constrained optimal values*.

Effects of a Constraint

- **Figure:** "Free extremum" vs. "Constrained extremum"



FINDING THE STATIONARY VALUES

Two Choice Variables with One Equality Constraint

➢ Use *elimination* (or *substitution*) method

Example: $$\max_{x_1, x_2} U = x_1 x_2 + 2x_1$$

subject to $4x_1 + 2x_2 = 60$

Two Choice Variables with One Constraint

➢Use *Lagrange-multiplier* method

Consider the following maximization problem.

$$\max_{x,y} f(x,y)$$

subject to $g(x,y)=c$.

Define the Lagrangian function by

F.O.C.

Example 1: 2 choice variables & 1 constraint

- Find the optimal values for the following function.

$$U = x_1 x_2 + 2x_1$$

subject to $4x_1 + 2x_2 = 60$

Step 1- Set up the Lagrangian function:

Step 2 - Find FONC:

Example 2: 2 choice variables & 1 constraint

- Find the optimal values for the following function:

$$f\left(x_1, x_2\right) = x_1^2 + x_2^2$$

subject to $g(x_1, x_2) = x_1 + 4x_2 = 2$

Step 1- Set up the Lagrangian function:

Step 2 - Find FONC:

Interpretation of Lagrange Multiplier

- Consider the problem

$$\max_{x,y}(\min) f(x,y)$$
$$\text{subject to } g(x,y)=c$$

- ➢ Solutions to this problem: *x*(c), y*(c),* and *f*(c).*
- ➢ *f*(c)* is called optimal value function.
- We can show that $\frac{df^*(c)}{dc}=\lambda(c)$
- ➢ The Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant, c.
- ➢ In economics, λ is called a "shadow price".

Interpretation of Lagrange Multiplier

- In economic applications, *c* denotes the available stock of some resources, *f(x, y)* denotes utility or profit.
- *λ(c)dc* is the change in utility or profit that can be obtained from *dc* units more of the resource.
- Thus, *λ* is called a shadow price of the resource.
- Examples:
  - For cost minimization problem, $\lambda = \frac{dC}{dQ_0} = MC$
  - For utility maximization problem, $\lambda = \frac{dU}{dY_0}$ = MU of income
  - For profit maximization problem, $\lambda = \frac{d\pi}{dQ_0}$
  - For output maximization problem, $\lambda = \frac{dQ}{dC_0}$

SECOND-ORDER CONDITIONS FOR CONSTRAINED OPTIMIZATION

Second-Order Total Differential

- From the constraint $g(x, y) = c$, $dg = g_x dx + g_y dy = 0$.

$$\Rightarrow \quad dy = -(g_x / g_y)dx$$

- From the objective function $z = f(x, y)$, $dz = f_x dx + f_y dy$.

$$d^2z =$$

➢

Second-Order Total Differential

- Recall the FONC for $L(x,y,\lambda) = f(x,y) + \lambda[c - g(x,y)]$

$$L_x(x,y) = f_x(x,y) - \lambda g_x(x,y) = 0$$

$$L_y(x,y) = f_y(x,y) - \lambda g_y(x,y) = 0$$

- By partially differentiating the FONC, the second derivatives are:

$$L_{xx}$$

$$L_{xy}$$

$$L_{yy}$$

➢ Thus, d^2z can be rewritten as:

$$d^2z =$$

Second-Order Conditions

- The second-order sufficient conditions are:
  - For *minimum* of z: *d^2z is positive definite*, subject to $dg = 0$
  - For *maximum* of z: *d^2z is negative definite*, subject to $dg = 0$

  where $dg = g_x dx + g_y dy = 0$.

- One can show the sign definiteness of d^2z can be determined from the following criterion:

$$d^2z \text{ is } \begin{Bmatrix} positve\_definite \\ negative\_definite \end{Bmatrix} \text{ subject to } g_x dx + g_y dy = 0$$

$$\text{iff } \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} \begin{cases} < 0 \\ > 0 \end{cases}$$

Second-Order Conditions

- The symmetric determinant $|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}$ is called the *Bordered Hessian*.

- Summary:

For a stationary value of $L(x, y) = f(x, y) + \lambda[c - g(x, y)]$

The second-order sufficient conditions are:

For *maximum* of z, the *bordered Hessian is positive* ($|\bar{H}| > 0$);

For *minimum* of z, the *bordered Hessian is negative* ($|\bar{H}| < 0$).

Example

- Determine whether the extremum of the objective function

$U = x_1 x_2 + 2x_1$ subject to $4x_1 + 2x_2 = 60$

gives a minimum or a maximum.

Example

- Determine whether the extremum of the objective function

$$f\left(x_1, x_2\right) = x_1^2 + x_2^2 \quad \text{subject to} \quad g(x_1, x_2) = x_1 + 4x_2 = 2$$

gives a minimum or a maximum.

ECONOMIC APPLICATIONS OF CONSTRAINED OPTIMIZATION

Utility Maximization Problem

- Objective function: $\max_{Q_1, Q_2} U = U(Q_1, Q_2)$
- Subject to: $Y_0 = p_1 Q_1 + p_2 Q_2$

➢ Lagrangian function:

Utility Maximization Problem

- Illustration of an Indifference Curve

Utility Maximization Problem

- Second-order sufficient condition

Utility Maximization Problem

- Example: Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$. Find A and B that maximize the utility given that $P_A = 10$, $P_B = 5$, and $Y = 490$.

Expenditure Minimization

- Objective function: $\underset{Q_1,Q_2}{Min} E = p_1 Q_1 + p_2 Q_2$
- Subject to: $\overline{U} = U(Q_1, Q_2)$

➢ Lagrangian function:

Expenditure Minimization

- Example: Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$, and the consumer would like to have a utility fixed at U_0. Find A and B that minimize the expenditure given that $P_A = 10$ and $P_B = 5$.

Profit Maximization Problem

- Objective function: $\max_{Q_1, Q_2} \pi = p_1 Q_1 + p_2 Q_2 - c(Q_1, Q_2)$
- Subject to: $Q_0 = Q_1 + Q_2$

➢ Lagrangian function:

Profit Maximization Problem

- Example: Suppose that P_1 = 80, P_2 = 100, $TC = 100 + 0.1Q_1^2 + 0.2Q_2^2$

And $Q_1 + Q_2 = 325$. Find Q_1 and Q_2 that maximize the profit.

Output Maximization Problem

- Objective function: $\max_{K,L} Q = Q(K,L)$
- Subject to: $C_0 = wL + rK$

➢ Lagrangian function:

Output Maximization Problem

- Example: Suppose $Q = KL$, w = 6, r = 10, C_0 = 60. Find K and L that maximizes output.

Cost Minimization

- Objective function: $\underset{K,L}{Min} C = wL + rK$
- Subject to: $Q_0 = f(K,L)$

➢ Lagrangian function:

Cost Minimization

- Graph: Expansion path

Cost Minimization

- Example: Suppose $Q = KL$. A firm will produce 15 units of the good. Find K and L that minimize total cost given that w = 6, r =10.

EXTENSION: N-CHOICE VARIABLE AND MULTI-CONSTRAINT CASES

Optimization with Equality Constraints:

n-Variable and One Equality Constraint

- The objective function

$$z = f\left(x_1, x_2, \ldots, x_n\right)$$

Subject to the constraint $g\left(x_1, x_2, \ldots, x_n\right) = c$

The Lagrangian function:

FONC:

Second-Order Conditions:
n-Variable and One Equality Constraint

- The objective function $z = f(x_1, x_2, \ldots, x_n)$

  Subject to $g(x_1, x_2, \ldots, x_n) = c$

➢ Bordered Hessian:

$$|\overline{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}$$

➢ Bordered leading principal minors can be defined as:

Second-Order Conditions: *n*-Variable and One Equality Constraint

- The conditions for positive and negative definiteness of d^2z are:

d^2z is $\begin{Bmatrix} positive\_definite \\ negative\_definite \end{Bmatrix}$ subject to $dg = 0$ iff $\begin{cases} |\bar{H}_2|, |\bar{H}_3|, \ldots, |\bar{H}_n| < 0 \\ |\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0; etc. \end{cases}$

Condition	Maximum	Minimum
First-order necessary condition	$L_\lambda = L_1 = L_2 = \ldots = L_n = 0$	$L_\lambda = L_1 = L_2 = \ldots = L_n = 0$
Second-order necessary condition*	$|\bar{H}_2| > 0; |\bar{H}_3| < 0;$ $|\bar{H}_4| > 0; \ldots; (-1)^n |\bar{H}_n| > 0$	$|\bar{H}_2|, |\bar{H}_3|, \ldots, |\bar{H}_n| < 0$

Cost Minimization

- Objective function: $\underset{K,L,T}{Min} C = wL + iK + rT$
- Subject to: $Q_0 = f(K,L,T)$

➢ Lagrangian function:

Optimization with Equality Constraints: *n*-Variable and Two Equality Constraints

- The objective function

$$z = f\left(x_1, x_2, \ldots, x_n\right)$$

Subject to the constraints $g\left(x_1, x_2, \ldots, x_n\right) = c$

$$h\left(x_1, x_2, \ldots, x_n\right) = d$$

- **The Lagrangian function:**

- **FONC:**

Optimization with Equality Constraints: *n*-Variable and *k* Equality Constraints

- The objective function

$$z = f(x_1, x_2, \ldots, x_n)$$

Subject to the constraints

$$g_1(x_1, x_2, \ldots, x_n) = c_1$$
$$g_2(x_1, x_2, \ldots, x_n) = c_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$g_k(x_1, x_2, \ldots, x_n) = c_k$$

➢ **The Lagrangian function:**

$$L = f(x_1, x_2, \ldots, x_n) + \lambda_1[c_1 - g_1(x_1, x_2, \ldots, x_n)] + \lambda_2[c_2 - g_2(x_1, x_2, \ldots, x_n)]$$
$$+ \ldots + \lambda_k[c_k - g_k(x_1, x_2, \ldots, x_n)]$$

Optimization with Equality Constraints:

n-Variable and *k* Equality Constraints

- **FONC:**

$$\frac{\partial L}{\partial x_1}$$

$$\frac{\partial L}{\partial x_2}$$

$$\dots$$

$$\frac{\partial L}{\partial x_n}$$

$$\frac{\partial L}{\partial \lambda_1}$$

$$\dots$$

$$\frac{\partial L}{\partial \lambda_n}$$

Second-Order Conditions:
n-Variable and *k* Equality Constraints

The Lagrangian condition is:

$$L = f(x_1, x_2, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j \left[c_j - g^j(x_1, x_2, \ldots, x_n) \right]$$

- Bordered Hessian:

$$\left|\overline{H}\right| \equiv \left| \begin{array}{cccc:cccc} 0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_1^k & g_2^k & \cdots & g_n^k \\ \hdashline g_1^1 & g_1^2 & \cdots & g_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^k & L_{21} & L_{22} & \cdots & L_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & g_n^2 & \cdots & g_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{array} \right|$$

Second-Order Conditions:
n-Variable and *k* Equality Constraints

- The second-order sufficient conditions are stated in terms of the signs of the following *(n-k)* bordered leading principal minors:

$$|\bar{H}_{k+1}|, |\bar{H}_{k+2}|, \ldots, |\bar{H}_n| \qquad \text{where} \quad |\bar{H}_n| = |\bar{H}|$$

- SOSC:

1) For a *maximum of* z, the bordered leading principal minors alternate in sign, and the sign of $|\bar{H}_{k+1}| = (-1)^{k+1}$.
2) For a *minimum of* z, all the bordered leading principal minors take the same sign, namely that of $(-1)^k$.

Example

- The objective function

$$z = f\left(x_1, x_2, x_3\right)$$

Subject to

$$g\left(x_1, x_2, x_3\right) = c$$
$$h\left(x_1, x_2, x_3\right) = d$$

➢ The Lagrangian function:

$$L = f\left(x_1, x_2, x_3\right) + \lambda_1\left[c - g\left(x_1, x_2, x_3\right)\right] + \lambda_2\left[d - h\left(x_1, x_2, x_3\right)\right]$$