

Macroeconomics

Lecture 1

Dynamic Problem

- **Max** $U_0(x_0, c_0) + U_1(x_1, c_1) + \dots + U_T(x_T, c_T)$
- **Subject to** (i) x_0 given,
- **and** (ii) the transition equations

$$x_1 = g_0(x_0, c_0),$$

$$x_2 = g_1(x_1, c_1),$$

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 $\vdots$

$$x_T = g_{T-1}(x_{T-1}, c_{T-1}),$$

$$x_{T+1} = 0 = g_T(x_T, c_T)$$

Dynamic Problem

- Assume that:
- (1) there is a time separable function, $U_t(x_t, c_t)$, which is a concave function.

Let x_t be a state variable at time t ,
 $t=0, 1, \dots, T$.

Let c_t be a control variable at time t ,
 $t=0, 1, \dots, T$.

Dynamic Problem

- (2) there is a transition function whose variables are defined by the set
- $\{x_{t+1}, x_t, c_t \mid x_{t+1} \leq g_t(x_t, c_t), c_t \in R\}$
- which is convex and compact set. (See definitions of these terms from Takayama, A. "Mathematical Economics" (1985), p.20 and p.32).
- Then

Dynamic Problem

- The values of $U_s(x_s, c_s)$ for $s \geq t$ are independent to the past values of c_v and x_v for $v < t$. This is the consequence of the time separable structure of (1.1) and (1.2)
- Forming the Lagrangian

$$L = \sum_{t=0}^T U_t(x_t, c_t) + \sum_{t=0}^T \lambda_t [g_t(x_t, c_t) - x_{t+1}], \quad (1.3)$$

where $x_{T+1} = 0$

- where λ_t is a Lagrange multipliers for $t = 0, 1, \dots, T$.

Two Methods

- A system of simultaneous equations,
- Backward Recursive method.

A system of simultaneous equations

Simultaneous Equations

$$L = \sum_{t=0}^T U_t(x_t, c_t) + \sum_{t=0}^T \lambda_t [g_t(x_t, c_t) - x_{t+1}], \quad (1.3)$$

where $x_{T+1} = 0$.

Note that, one can set

$$U_t(x_t, c_t) = x_t \cdot U(c_t),$$

where $x_t = (1+r)^{-t}$ = discount factor.

Simultaneous Equations

$$\begin{aligned} \frac{\partial L}{\partial c_t} = 0, \quad \frac{\partial U_t(x_t, c_t)}{\partial c_t} + \frac{\partial g_t(x_t, c_t)}{\partial c_t} \lambda_t = 0, \quad t = 0, \dots, T-1 \\ \frac{\partial L}{\partial c_T} = \frac{\partial U_T(x_T, c_T)}{\partial c_T} \geq 0. \end{aligned} \quad (1.4a)$$

$$\begin{aligned} \frac{\partial L}{\partial \lambda_t} = 0, \quad x_{t+1} = g_t(x_t, c_t), \quad t = 0, 1, \dots, T-1, \\ x_{T+1} = 0 = g_T(x_T, c_T) \end{aligned} \quad (1.4b)$$

Simultaneous Equations

- There are $2(T+1)$ unknowns :
 - c_t for $t = 0, 1, 2, \dots, T$,
 - and λ_t for $t = 0, 1, 2, \dots, T$.)
- There are $2(T+1)$ equations.
- Unique optimal solution exists for each unknown.

Simultaneous Equations

- Optimal policies

$$c_0^* = h_0(x_0, x_1, \dots, x_T),$$

$$c_1^* = h_1(x_0, x_1, \dots, x_T),$$

$$c_2^* = h_2(x_0, x_1, \dots, x_T),$$

$$\lambda_0^* = k_0(x_0, x_1, \dots, x_T),$$

$$\lambda_1^* = k_1(x_0, x_1, \dots, x_T),$$

$$\lambda_2^* = k_2(x_0, x_1, \dots, x_T),$$

•
•
•

(1.4c)

$$c_T^* = h_T(x_0, x_1, \dots, x_T),$$

$$\lambda_{T-1}^* = k_{T-1}(x_0, x_1, \dots, x_T),$$

and $g_T(x_T, c_T) = 0 = x_{T+1}$

Backward recursive method

Recursive method

$$\frac{\partial L}{\partial c_t} = \frac{\partial U_t(x_t, c_t)}{\partial c_t} + \frac{\partial g_t(x_t, c_t)}{\partial c_t} \lambda_t = 0, \quad t = 0, \dots, T-1$$

$$\frac{\partial L}{\partial c_T} = \frac{\partial U_T(x_T, c_T)}{\partial c_T} \geq 0$$

(1.4d)

$$\frac{\partial L}{\partial x_t} = \frac{\partial U_t(x_t, c_t)}{\partial x_t} + \frac{\partial g_t(x_t, c_t)}{\partial x_t} \lambda_t - \lambda_{t-1} = 0, \quad t = 1, \dots, T-1$$

$$\frac{\partial L}{\partial x_T} = \frac{\partial U_T(x_T, c_T)}{\partial x_T} - \lambda_{T-1} = 0,$$

(1.4e)

$$\frac{\partial L}{\partial \lambda_t} = 0, \quad x_{t+1} = g_t(x_t, c_t), \quad t = 0, 1, \dots, T-1,$$

$$x_{T+1} = 0 = g_T(x_T, c_T).$$

(1.4f)

Recursive method

- From (1.4e), one can have

$$\lambda_t = \frac{\partial U_{t+1}(x_{t+1}, c_{t+1})}{\partial x_{t+1}} + \frac{\partial g_{t+1}(x_{t+1}, c_{t+1})}{\partial x_{t+1}} \lambda_{t+1}$$

- Using this and (1.4d) recursively to eliminate λ_t , $t=0,1,\dots,T-1$.
- From (1.4d) and (1.4f), one has

$$\frac{\partial U_t(x_t, c_t)}{\partial c_t} + \frac{\partial g_t(x_t, c_t)}{\partial c_t} \left\{ \frac{\partial U_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}} \left[\frac{\partial U_{t+2}}{\partial x_{t+2}} + \frac{\partial g_{t+2}}{\partial x_{t+2}} \left\{ \frac{\partial U_{t+3}}{\partial x_{t+3}} + \frac{\partial g_{t+3}}{\partial x_{t+3}} \left\{ \dots + \frac{\partial g_{T-1}}{\partial x_{T-1}} \left[\frac{\partial U_T}{\partial x_T} \right] \dots \right\} \right] \right] \right\} = 0,$$

for $t = 0, 1, \dots, T$.

(1.5a)

Recursive method

- $x_{t+1} = g_t(x_t, c_t), \quad t = 0, 1, \dots, T-1. \quad (1.5b)$

- $x_{T+1} = 0 = g_T(x_T, c_T). \quad (1.5c)$

- At $t = T$, we have from Eq. (1.4d), (1.5a), (1.5b), (1.5c), that

- $\frac{\partial U_T(x_T, c_T)}{\partial c_T} \geq 0 \quad (1.5d)$

- $x_{T+1} = 0 = g_T(x_T, c_T) \quad (1.5e)$

Recursive method

- Now, consider the following optimization problem at T

$$\max \quad U_T(x_T, c_T) \quad (1.1a)$$

$$s.t. \quad x_T \text{ given,}$$

$$x_{T+1} = g_T(x_T, c_T) = 0 \quad (1.2a)$$

- It should be pointed out that this problem is a sub-problem of problem (1.1) and (1.2)

Lagrangian function

$$L_T = U_T(x_T, c_T) + \lambda_T [0], \quad \text{given } x_T$$

$$\frac{\partial L_T}{\partial c_T} = \frac{\partial U_T(x_T, c_T)}{\partial c_T} \geq 0$$

Recursive method

- For $t = T-1$, the optimization problem at $T-1$ is

$$\max \quad U_{T-1}(x_{T-1}, c_{T-1}) + U_T(x_T, c_T), \quad (1.1aa)$$

$$s.t. \quad x_{T-1} \text{ given},$$

$$\left. \begin{aligned} x_T &= g_{T-1}(x_{T-1}, c_{T-1}), \\ x_{T+1} &= 0 = g_T(x_T, c_T). \end{aligned} \right\} \quad (1.2aa)$$

- This problem is, again, a sub-problem of problem (1.1) and (1.2)

Recursive method

- And the 1st-order conditions are

$$\frac{\partial L}{\partial c_{T-1}} = \frac{\partial U_{T-1}(x_{T-1}, c_{T-1})}{\partial c_{T-1}} + \frac{\partial g_{T-1}(x_{T-1}, c_{T-1})}{\partial c_{T-1}} \lambda_{T-1} = 0, \quad (1.4aaa)$$

$$\frac{\partial L}{\partial c_T} = \frac{\partial U_T(x_T, c_T)}{\partial c_T} \geq 0$$

$$\frac{\partial L}{\partial x_{T-1}} = \frac{\partial U_{T-1}(x_{T-1}, c_{T-1})}{\partial x_{T-1}} + \frac{\partial g_{T-1}(x_{T-1}, c_{T-1})}{\partial x_{T-1}} \lambda_{T-1} - \lambda_{T-2} = 0, \quad (1.4bbb)$$

$$\frac{\partial L}{\partial x_T} = \frac{\partial U_T(x_T, c_T)}{\partial x_T} - \lambda_{T-1} = 0 \quad (1.4ccc)$$

•

$$\begin{aligned} x_T &= g_{T-1}(x_{T-1}, c_{T-1}), \\ x_{T+1} &= 0 = g_T(x_T, c_T) \end{aligned} \quad (1.4ddd)$$

Recursive method

- By using (1.4aaa),(1.4bbb) and (1.4ccc), we have, at $t=T-1$, that

$$\frac{\partial U_{T-1}(x_{T-1}, c_{T-1})}{\partial c_{T-1}} + \frac{\partial g_{T-1}(x_{T-1}, c_{T-1})}{\partial c_{T-1}} \left[\frac{\partial U_T(x_T, c_T)}{\partial x_T} \right] = 0 \quad (1.5ccc)$$

Equation (1.5ccc) can also be obtained from (1.5a), given that $t=T-1$. Then recall (1.4ddd)

$$x_T = g_{T-1}(x_{T-1}, c_{T-1}). \quad (1.5ddd)$$

Recursive method

- Next, use $h_T(x_T)$ and $g_{T-1}(x_{T-1}, h(x_{T-1}))$ to replace c_T and X_T in (1.5ccc), we then have

$$\frac{\partial U_{T-1}(x_{T-1}, c_{T-1})}{\partial c_{T-1}} + \frac{\partial g_{T-1}(x_{T-1}, c_{T-1})}{\partial c_{T-1}} \left[\frac{\partial U_T(x_T, h_T(x_T))}{\partial x_T} \right] = 0, \quad (1.5ccc *)$$

$$x_T = g_{T-1}(x_{T-1}, c_{T-1}), \quad (1.5ddd *)$$

Recursive method

- Given x_{T-1} , equations (1.5ccc*) and equations (1.5ddd*) form a system of 2 equations in (x_T, c_{T-1}) .
- Solving these 2 equations for x_T and c_{T-1} as functions of x_{T-1} :
- $x_T = f_{T-1}(x_{T-1}),$
- $c_{T-1} = h_{T-1}(x_{T-1})$