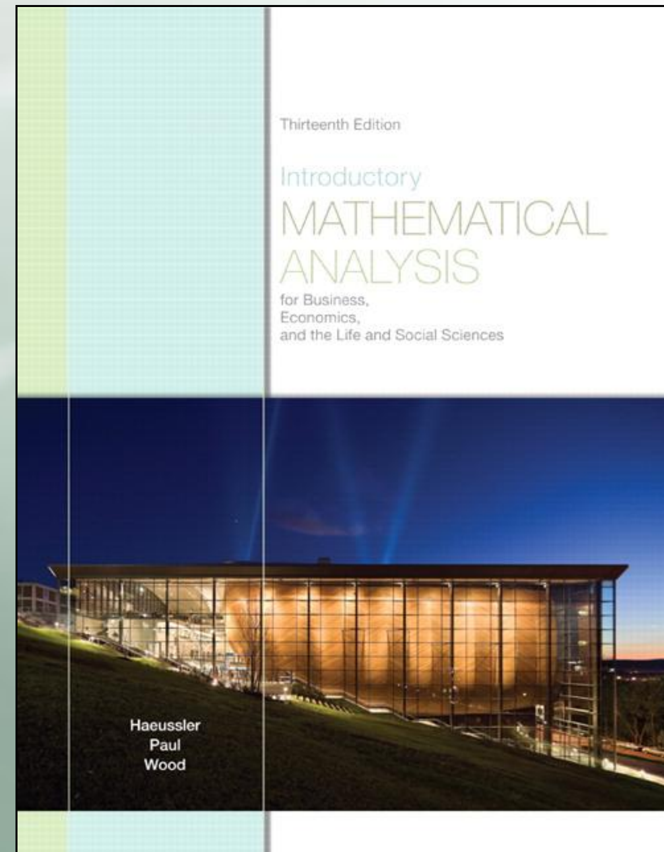
INTRODUCTORY MATHEMATICAL ANALYSIS

For Business, Economics, and the Life and Social Sciences

Chapter 1
Applications and More Algebra



Chapter Objectives

- To model situations described by linear or quadratic equations.
- To solve linear inequalities in one variable and to introduce interval notation.
- To model real-life situations in terms of inequalities.
- To solve equations and inequalities involving absolute values.
- To write sums in summation notation and evaluate such sums.

Chapter Outline

1.1) Applications of Equations

1.2) Linear Inequalities

1.3) Applications of Inequalities

1.4) Absolute Value

1.5) Summation Notation

1.1 Applications of Equations

- **Modeling:** Translating relationships in the problems to mathematical symbols.

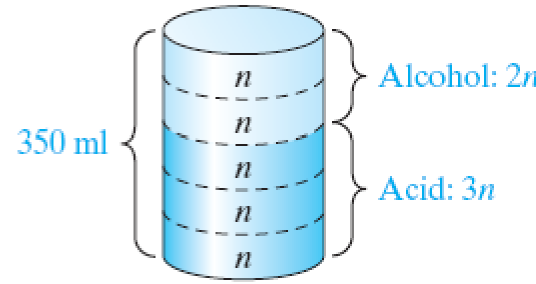
Example 1 - Mixture

A chemist must prepare 350 ml *of a chemical solution made up of two parts alcohol and three parts acid. How much of each should be used?*

Solution:

Let n = number of milliliters in each part.

$$2n + 3n = 350$$
$$5n = 350$$
$$n = \frac{350}{5} = 70$$



Each part has 70 ml.

Amount of alcohol = 2n = 2(70) = 140 ml

Amount of acid = 3n = 3(70) = 210 ml

- **Fixed cost** is the sum of all costs that are independent of the level of production.
- **Variable cost** is the sum of all costs that are dependent on the level of output.
- **Total cost** = variable cost + fixed cost
- **Total revenue** = (price per unit) x (number of units sold)
- **Profit** = total revenue − total cost

Example 3 – Profit

The Anderson Company produces a product for which the variable cost per unit is \$6 *and the fixed cost is* \$80,000. *Each unit has a selling price of* \$10. *Determine the number of units that must be sold for the company to earn a profit of* \$60,000.

Solution:

Let q = number of sold units.

variable cost = $6q$

total cost = $6q + 80{,}000$

total revenue = $10q$

Since profit = total revenue − total cost

$$60{,}000 = 10q - (6q + 80{,}000)$$
$$140{,}000 = 4q$$
$$35{,}000 = q$$

35,000 units must be sold to earn a profit of $60,000.

Example 5 – Investment

A total of \$10,000 *was invested in two business ventures, A and B. At the end of the first year, A and B yielded returns of* 6%*and* 5.75 %, *respectively, on the original investments. How was the original amount allocated if the total amount earned was* \$588.75?

Example 5 – Investment

Solution:

Let x = amount ($) invested at 6%.

$$\begin{aligned}(0.06)x+(0.0575)(10{,}000-x)&=588.75\\ 0.06x+575-0.0575x&=588.75\\ 0.0025x&=13.75\\ x&=5500\end{aligned}$$

$5500 was invested at 6%

$10,000−$5500 = $4500 was invested at 5.75%.

Example 7 – Apartment Rent

A real-estate firm owns the Parklane Garden Apartments, which consist of 96 *apartments. At* $550 *per month, every apartment can be rented. However, for each* $25 *per month increase, there will be three vacancies with no possibility of filling them. The firm wants to receive* $54,600 *per month from rent. What rent should be charged for each apartment?*

Solution 1:

Let r = rent ($) to be charged per apartment.

Total rent = (rent per apartment) x (number of apartments rented)

Solution 1 (Con't):

$$54{,}600 = r\left[96 - \frac{3(r-550)}{25}\right]$$

$$54{,}600 = r\left[\frac{2400 - 3r + 1650}{25}\right]$$

$$54{,}600 = r\left[\frac{4050 - 3r}{25}\right]$$

$$1{,}365{,}000 = r(4050 - 3r)$$

$$3r^2 - 4050r + 1{,}365{,}000 = 0$$

$$r = \frac{4050 \pm \sqrt{(-4050)^2 - 4(3)(1{,}365{,}000)}}{2(3)}$$

$$= \frac{4050 \pm \sqrt{22{,}500}}{6} = 675 \pm 25$$

Rent should be \$650 or \$700.

Solution 2:

Let n = number of $25 increases.

Total rent = (rent per apartment) x
(number of apartments rented)

Solution 2 (Con’t):

$$54{,}600 = (550 + 25n)(96 - 3n)$$

$$54{,}600 = 52{,}800 + 750n - 75n^2$$

$$75n^2 - 750n + 1800 = 0$$

$$n^2 - 10n + 24 = 0$$

$$(n - 6)(n - 4) = 0$$

$$n = 6 \text{ or } 4$$

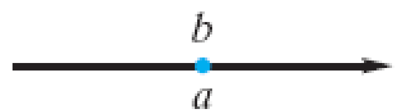
The rent charged should be either

550 + 25(6) = \$700 or

550 + 25(4) = \$650.

1.2 Linear Inequalities

- Supposing *a* and *b* are two points on the real-number line, the relative positions of two points are as follows:

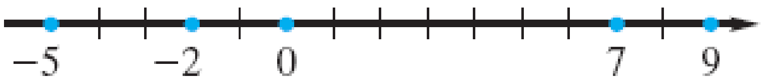


$a = b$

$a < b$, a is less than b
$b > a$, b is greater than a

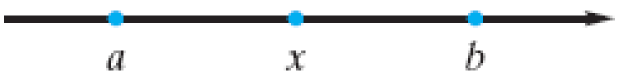
$a > b$, a is greater than b
$b < a$, b is less than a

- We use dots to indicate points on a number line.



- Suppose that *a* < *b* and *x* is between *a* and *b*.

$$\boxed{a < x < b}$$



- **Inequality** is a statement that one number is less than another number.

- Rules for Inequalities:

1. If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
2. If $a < b$ and $c > 0$, then $ac < bc$ and $a/c < b/c$.
3. If $a < b$ and $c < 0$, then $a(c) > b(c)$ and $a/c > b/c$.
4. If $a < b$ and $a = c$, then $c < b$.
5. If $0 < a < b$ or $a < b < 0$, then $1/a > 1/b$.
6. If $0 < a < b$ and $n > 0$, then $a^n < b^n$.

   If $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

- **Linear inequality** can be written in the form

$$ax + b < 0$$

where a and b are constants and $a \neq 0$

- To **solve** an inequality involving a variable is to find all values of the variable for which the inequality is true.

Example 1 – Solving a Linear Inequality

Solve $2(x - 3) < 4$.

Solution:
Replace inequality by equivalent inequalities.

$$2(x-3) < 4$$
$$2x - 6 < 4$$
$$2x - 6 + 6 < 4 + 6$$
$$2x < 10$$
$$\frac{2x}{2} < \frac{10}{2}$$
$$x < 5$$

Example 3 – Solving a Linear Inequality

Solve (3/2)(s − 2) + 1 > −2(s − 4).

Solution:

$$\frac{3}{2}(s-2)+1>-2(s-4)$$

$$2\left[\frac{3}{2}(s-2)+1\right]>2[-2(s-4)]$$

$$2[3(s-2)+2]>-4(s-4)$$

$$3s-4>-4s+16$$

$$7s>20$$

$$s>\frac{20}{7}$$

The solution is (20/7 ,∞).

1.3 Applications of Inequalities

- Solving word problems may involve inequalities.

Example 1 - Profit

For a company that manufactures aquarium heaters, the combined cost for labor and material is $21 *per heater. Fixed costs (costs incurred in a given period, regardless of output) are* $70,000. *If the selling price of a heater is* $35, *how many must be sold for the company to earn a profit?*

Solution:

profit = total revenue − total cost

Let q = number of heaters sold.

$$\begin{aligned} \text{total revenue} - \text{total cost} &> 0 \\ 35q - (21q + 70{,}000) &> 0 \\ 14q &> 70{,}000 \\ q &> 5000 \end{aligned}$$

Example 3 – Current Ratio

After consulting with the comptroller, the president of the Ace Sports Equipment Company decides to take out a short-term loan to build up inventory. The company has current assets of \$350,000 *and current liabilities of* \$80,000. *How much can the company borrow if the current ratio is to be no less than* 2.5*? (Note: The funds received are considered as current assets and the loan as a current liability.)*

Solution:

Let x = amount the company can borrow.

Current ratio = Current assets / Current liabilities

We want,

$$\frac{350{,}000 + x}{80{,}000 + x} \geq 2.5$$

$$350{,}000 + x \geq 2.5(80{,}000 + x)$$

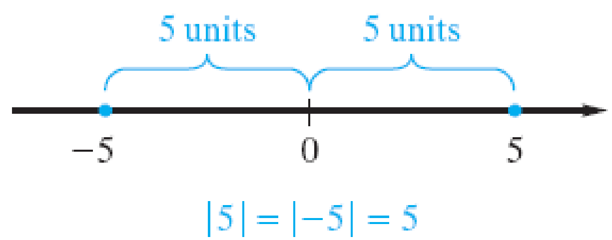
$$150{,}000 \geq 1.5x$$

$$100{,}000 \geq x$$

The company may borrow up to $100,000.

1.4 Absolute Value

- On real-number line, the distance of *x* from 0 is called the **absolute value** of *x*, denoted as |*x*|.



DEFINITION

The **absolute value** of a real number *x*, written |*x*|, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Example 1 – Solving Absolute-Value Equations

a. *Solve* $|x - 3| = 2$

b. *Solve* $|7 - 3x| = 5$

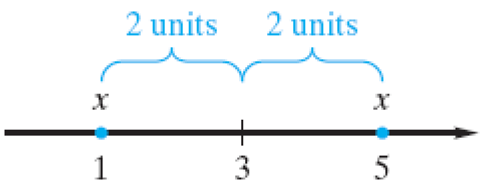
c. *Solve* $|x - 4| = -3$

Example 1 – Solving Absolute-Value Equations

Solution:

a. $x - 3 = 2$ or $x - 3 = -2$

$x = 5$ $\quad$ $x = 1$



b. $7 - 3x = 5$ or $7 - 3x = -5$

$x = 2/3$ $\quad$ $x = 4$

c. The absolute value of a number is never negative. The solution set is $\varnothing$.

Absolute-Value Inequalities

- Summary of the solutions to absolute-value inequalities is given.

Inequality ($d > 0$)	Solution
$\lvert x \rvert < d$	$-d < x < d$
$\lvert x \rvert \leq d$	$-d \leq x \leq d$
$\lvert x \rvert > d$	$x < -d$ or $x > d$
$\lvert x \rvert \geq d$	$x \leq -d$ or $x \geq d$

Example 3 – Solving Absolute-Value Equations

a. *Solve* $|x + 5| \geq 7$

b. *Solve* $|3x - 4| > 1$

Solution:

a. $x + 5 \leq -7$ or $x + 5 \geq -7$

$x \leq -12$ $\qquad$ $x \geq 2$

We write it as $(-\infty, -12] \cup [2, \infty)$, where $\cup$ is the *union* symbol.

b. $3x - 4 < -1$ or $3x - 4 > 1$

$x < 1$ $\qquad$ $x > \frac{5}{3}$

We can write it as $(-\infty, 1) \cup \left(\frac{5}{3}, \infty\right)$.

Properties of the Absolute Value

- 5 basic properties of the absolute value:

1. $|ab| = |a| \cdot |b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
3. $|a - b| = |b - a|$
4. $-|a| \le a \le |a|$
5. $|a + b| \le |a| + |b|$

- Property 5 is known as *the triangle inequality*.

Example 5 – Properties of Absolute Value

Solution:

a. $\left|(-7)\cdot 3\right| = |-7|\cdot|3| = 21$

b. $|4-2| = |2-4| = 2$

c. $|7-x| = |x-7|$

d. $\left|\frac{-7}{3}\right| = \frac{|-7|}{|3|} = \frac{7}{3};\ \left|\frac{-7}{-3}\right| = \frac{|-7|}{|-3|} = \frac{7}{3}$

e. $\left|\frac{x-3}{-5}\right| = \frac{|x-3|}{|-5|} = \frac{|x-3|}{5}$

f. $-|2| \le 2 \le |2|$

g. $\left|(-2)+3\right| = |1| = 1 \le 5 = 2+3 = |-2|+|3|$

1.5 Summation Notation

DEFINITION

The sum of the numbers a_i, with i successively taking on the values m through n is denoted as

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

Example 1 – Evaluating Sums

Evaluate the given sums.

a. $\sum_{n=3}^{7}(5n-2)$

b. $\sum_{j=1}^{6}\left(j^2+1\right)$

Solution:

a. $\sum_{n=3}^{7}(5n-2)=[5(3)-2]+[5(4)-2]+[5(5)-2]+[5(6)-2]+[5(7)-2]$

$$=13+18+23+28+33$$

$$=115$$

b. $\sum_{j=1}^{6}\left(j^2+1\right)=\left(1^2+1\right)+\left(2^2+1\right)+\left(3^2+1\right)+\left(4^2+1\right)+\left(5^2+1\right)+\left(6^2+1\right)$

$$=2+5+10+17+26+37$$

$$=97$$

- To sum up consecutive numbers, we have

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

where *n* = the last number.

Example 3 – Applying the Properties of Summation Notation

Evaluate the given sums.

a. $\sum_{j=30}^{100} 4$ b. $\sum_{k=1}^{100} (5k+3)$ c. $\sum_{k=1}^{200} 9k^2$

Solution:

a. $$\sum_{j=30}^{100} 4 = \sum_{i=1}^{71} 4 = 4 \cdot 71 = 284$$

b. $$\sum_{k=1}^{100} (5k+3) = \sum_{k=1}^{100} 5k + \sum_{k=1}^{100} 3 = 5\left(\frac{100 \cdot 101}{2}\right) + 3(100) = 25{,}550$$

c. $$\sum_{k=1}^{200} 9k^2 = 9\sum_{k=1}^{200} k^2 = 9\left(\frac{200 \cdot 201 \cdot 401}{6}\right) = 24{,}180{,}300$$