- Echelon form
- Reduced echelon form
- The break down of elimination
- Rectangular matrices
- Complete solutions to <u>Ax</u>=<u>b</u>

Elimination ⇨ Simplify a system of linear equations without changing any of the solutions.

Elementary Row Operations:

1. Replacement (multiply one row and subtract it from another row)
2. Interchange
3. Scaling (multiply all entries in a row by a nonzero constant)

Each of these operation is reversible and does not change the solution set.

Good matrices➔ elimination work ➔ get a solution in an efficient way

When elimination fail to give solutions?

How the elimination decide whether a system is good or has problem?

Echelon form (or **row echelon form**):

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

EXAMPLE: Echelon forms

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$

The whole purpose of doing the elimination is to get an upper triangular matrix (<u>U</u>) (echelon form)

Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

Important terms

Pivot position: a position of a leading entry in an echelon form of the matrix.

Pivot: a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's

Pivot column: a column that contains a pivot position.

Pivot row: a row that contains a pivot position.

Example : 3 equations 3 unknowns

(coefficient matrix)

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{2} & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

The first pivot

Pivot row

Pivot column

Replacement

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 8 & 3 \end{bmatrix} \xrightarrow{\text{Replacement}} \begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

U

By definition, pivots cannot be zero. (need to divide them!)

Example Row reduce to echelon form and locate the pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot

pivot column

Possible Pivots:

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Original Matrix:

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

pivot columns: 1 2 4

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

EXAMPLE: Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Echelon form

Reduced echelon form

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 2 & 6 & 8\\ 2 & 8 & 10\end{bmatrix}$$

How many pivot variables?

Under what circumstances could the elimination stop?

If the algorithm produces ***n*** pivots (there are pivots in every column i.e. a full set of pivots), then there is only one solution to the equations (a unique solution)

example

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned} \iff \underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

If a **zero** appears in a pivot position, elimination has to stop!

Stop temporarily➔ there is possibility to exchange with a lower row for a proper pivot.

Stop permanently➔ there is no exchange of row that can avoid zero.

examples

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned} \iff \underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned} \iff \underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

However, we do not know whether a zero will appear until we try.

How could this 3x3 case fail to give a unique solution?

$$\begin{bmatrix}\circledast & * & *\\ * & * & *\\ * & * & *\end{bmatrix}$$

-Fail to come up with 3 pivots.

Example

$$\begin{bmatrix}1 & 2 & 1\\ 3 & 6 & 1\\ 0 & 4 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1\\ 0 & \textcircled{0} & -2\\ 0 & 4 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1\\ 0 & 4 & 1\\ 0 & 0 & -2\end{bmatrix}$$

If this entry is changed to 0

EXAMPLE: Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$\begin{bmatrix}1 & -2 & 1\\ 0 & 2 & -8\\ -4 & 5 & 9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 8\\ -9\end{bmatrix} \iff \left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & 2 & -8 & 8\\ -4 & 5 & 9 & -9\end{array}\right]$$

Perform row operation to obtain echelon form

In the last example, this system was reduced to the triangular form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c}1 & -2 & 1 & 0\\ 0 & 1 & -4 & 4\\ 0 & 0 & 1 & 3\end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

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EXAMPLE: Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \quad \left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

Solution: Row operations produce:

$$\left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \end{aligned} \quad \leftarrow \textit{Never true}$$

The original system is inconsistent!

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EXAMPLE: For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

Solution: Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

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Homogeneous linear systems

<u>Ax</u>=<u>0</u>

E.g.

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 3 & 0 \end{array}\right] \overset{R_2-\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 1 & 0 \end{array}\right] \quad \therefore x_2 = 0 \text{ and } x_1 = 0 \quad \text{or} \quad \underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

E.g.

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \overset{R_2-\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \therefore x_2 \in R \text{ and } 0 = 2x_1 + 4x_2 \quad \text{or} \quad \underline{x} = C\begin{bmatrix} -2 \\ 1 \end{bmatrix}; C \in R$$

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Nonhomogeneous linear systems

<u>Ax</u>=<u>b</u>

e.g.

(a)
$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 4 \\ 1 & 3 & 1 \end{array}\right] \overset{R_2-\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 4 \\ 0 & 1 & -1 \end{array}\right] \quad \therefore x_2 = -1 \text{ and } x_1 = 4$$

(b)
$$\left[\begin{array}{cc|c} 2 & 4 & 4 \\ 1 & 2 & 1 \end{array}\right] \overset{R_2-\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 4 \\ 0 & 0 & -1 \end{array}\right]$$
No solution

(c)
$$\left[\begin{array}{cc|c} 2 & 4 & 4 \\ 1 & 2 & 2 \end{array}\right] \overset{R_2-\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 4 \\ 0 & 0 & 0 \end{array}\right]$$
Infinitely many solutions

$\therefore x_2 \in R$ and $x_1 = 2 - 2x_2$ or $\underline{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + C\begin{bmatrix} -2 \\ 1 \end{bmatrix}; C \in R$

Example

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

General solutions

$$x_1 = -1 + \frac{4}{3}x_3$$

$$x_2 = 2$$

As a vector the general solutions of Ax=b has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+\frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

x_3 can be any real number.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + C\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \quad C \in R$$

Rectangular (coefficient) matrices $\underline{\mathbf{A}}_{m\times n}$

$$\underline{\mathbf{A}}_{m\times n}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

This matrix equation is equivalent to a system of linear equations of *m* equations *n* unknowns.

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ x_1 + 5x_2 &= 6 \\ 2x_1 + 7x_2 &= 8 \\ 4x_1 + 14x_2 &= 16 \end{aligned} \iff \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 2 & 7 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 8 \\ 16 \end{bmatrix} \iff \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 4 & 14 & 16 \end{array}\right]$$

The solution of *m* equations *n* unknowns $(m \neq n)$

Elimination ➔ similar to square matrix
Back substitution➔ some differences

Rectangular (coefficient) matrices

3 equations 4 unknowns➔ $\underline{\mathbf{A}}\underline{\mathbf{x}}=\mathbf{0}$

$$A = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \xrightarrow[\text{row operation}]{\text{Perform}} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Candidate for the 2nd pivot has become zero and
The entry below is also zero. " go onto the next column"

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftarrow$$

U (Echelon form)

Ux = 0

Row of zero occurs because of row 3 was a combination of row 1 and 2

Number of pivots = 2 (rank of the matrix)
Solution is not unique

- Solution to <u>Ux</u>=0 is the same as solution to <u>Ax</u>=0

$$\begin{bmatrix} \textcircled{1} & 2 & 2 & 2 \\ 0 & 0 & \textcircled{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

- Number of pivot variables (any variables that corresponds to a pivot column) = r
- Number of free variables = n-r

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 0$$

- *Assign any numbers to X_2 and X_4*
- *Solve for X_1 and X_3*

⇨ $x_1 = -2x_2 - 2x_3 - 2x_4$, $x_3 = -2x_4$ Parametric description of general solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \; ; C_1, C_2 \in R$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = C_1\begin{bmatrix} -2 \\ \textcircled{1} \\ 0 \\ \textcircled{0} \end{bmatrix} + C_2\begin{bmatrix} 2 \\ \textcircled{0} \\ -2 \\ \textcircled{1} \end{bmatrix} \; ; C_1, C_2 \in R$$

General solutions as a vector

4 equations 2 unknowns➔ **<u>Ax</u>=0**

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 2 & 7 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Rectangular (coefficient) matrices $\underline{\mathbf{A}}_{m\times n}$

m<n → <u>No unique solution exists.</u> There may be

- infinitely many solutions
- no solution

(if the system in echelon form contains equations of the form 0=b with b nonzero.)

e.g.

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & -b_2 + 2b_1 + b_3 - 3b_1 \end{array}\right]$$

m>n → There may be

- A unique solution could exist provided that number of rank equal number of unknowns and m-n row are linear combination of other rows.
- infinitely many solutions
- no solutions

Example

a) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 4 & 14 & 16 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Unique solutions

m-n=2

These 2 rows are linear combination of other rows

(b) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 8 \\ 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & -5 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 5/3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5/3 \\ 0 & 0 & 0 \end{array}\right]$ No solutions

(c) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 6 \\ 5 & 10 & 10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ Many solutions

Elimination upward → Row Reduced Echelon Form

(zero above and below pivot)

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

We have got the identity matrix in RREF

Pivot=1

Pivot row 1&2

Pivot column 1&3

$\underline{\mathbf{Ax}}=0$

$\underline{\mathbf{Ux}}=0$

$\underline{\mathbf{Rx}}=0$

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \underline{\mathbf{R}}$$

← **Row reduced Echelon Form**

In pivot column

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \end{bmatrix}$$

Identity part I

In free column

$$\begin{bmatrix} 2 & -2 \\ 0 & 2 \\ \hdashline 0 & 0 \end{bmatrix}$$

Free part F

$$\underline{\mathbf{R}} = \begin{bmatrix} \underline{\mathbf{I}} & \underline{\mathbf{F}} \\ 0 & 0 \end{bmatrix}$$

Solve $\underline{\mathbf{Rx}}=\mathbf{0}$

Null space matrix (columns=special solutions)

$\underline{\mathbf{RN}}=\mathbf{0}$ $\quad \underline{\mathbf{N}} = \begin{bmatrix} -\underline{\mathbf{F}} \\ \underline{\mathbf{I}} \end{bmatrix}$

$$\begin{bmatrix} \underline{\mathbf{I}} & \underline{\mathbf{F}} \end{bmatrix} \begin{bmatrix} x_{\text{pivot}} \\ x_{\text{free}} \end{bmatrix} = 0$$

$$\underline{\mathbf{I}}x_{\text{pivot}} + \underline{\mathbf{F}}x_{\text{free}} = 0$$

When a nonhomogeneous linear system has many solutions,
The general solution can be written in parametric vector form as one vector plus an arbitrary linear combinations of vectors that satisfy the corresponding homogeneous system.

Suppose the equations $\underline{\mathbf{Ax}}=\underline{\mathbf{b}}$ is consistent for some given $\underline{\mathbf{b}}$ and let $\underline{\mathbf{X}}_p$ be a solution then the solutions set of $\underline{\mathbf{Ax}}=\underline{\mathbf{b}}$ is the set of all vectors of the form $\underline{\mathbf{X}}=\underline{\mathbf{X}}_p+\underline{\mathbf{X}}_n$ where $\underline{\mathbf{X}}_n$ is any solutions of the homogeneous system.

Example

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 2 & 4 & 6 & 8 & 5 \\ 3 & 6 & 8 & 10 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

General solutions as a vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} ; C_1, C_2 \in R$$

$\underline{\mathbf{X}} = \underline{\mathbf{X}}_p + \underline{\mathbf{X}}_n$

$\underline{\mathbf{Ax}}=\underline{\mathbf{b}}$ solvable if a combination of rows of $\underline{A}$ gives zero row then the same combination of the entries of $\underline{b}$ must give zero.

m×n matrix of rank r ➔ m rows
r pivots

r ≤ m

r ≤ n

Full column rank ➔ r = n (no free variables)
➔solution to $\underline{\mathbf{Ax}}=\underline{\mathbf{b}}$ is X=Xparticular
➔ $N(\mathbf{A})=\{0\}$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \quad \& \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}$$

RREF

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 6 & 1 & 7 \\ 5 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -5 & -5 \\ 0 & -17 & -17 \\ 0 & -14 & -14 \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

r=n <m
(full column rank)

Complete solutions to $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$

$$\underline{\mathbf{X}}_c = \underline{\mathbf{X}}_p + \underline{\mathbf{X}}_n$$

1. $\underline{\mathbf{X}}_p$: set free variables = 0
solve $\underline{\mathbf{Ax}}=\underline{\mathbf{b}}$ for pivot variables

2. $\underline{\mathbf{X}}_n$: $\underline{\mathbf{Ax}}=\underline{\mathbf{0}}$

$$\underline{\mathbf{A}}\underline{\mathbf{x}}_p = \underline{\mathbf{b}} \quad \textbf{and} \quad \underline{\mathbf{A}}\underline{\mathbf{x}}_\mathbf{n} = \underline{\mathbf{0}}$$
$$\underline{\mathbf{A}}(\underline{\mathbf{x}}_\mathbf{p} + \underline{\mathbf{x}}_\mathbf{n}) = \underline{\mathbf{b}} + \underline{\mathbf{0}}$$

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EXAMPLE:

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \qquad \begin{array}{l} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 8x_4 = 5 \\ x_5 = 7 \end{array}$$

Pivot columns:

Pivot variables: **Free variables:**

$$\begin{array}{l} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 8x_4 = 5 \\ x_5 = 7 \end{array} \qquad \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} + C_1 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}$$

(general solution) $C_1, C_2 \in R$

The system is consistent and has infinite many solutions

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EXAMPLE:

$$\begin{array}{l} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{array}$$

Perform sequences of row operation

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

Free variables: x_3 and x_4

Consistent system with free variables $\Rightarrow$ **infinitely many solutions.**

What is general solutions in a vector form?

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EXAMPLE:

$$\begin{array}{l} 3x_1 + 4x_2 = -3 \\ 2x_1 + 5x_2 = 5 \\ -2x_1 - 3x_2 = 1 \end{array} \rightarrow \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

Consistent system, no free variables $\Rightarrow$ **unique solution.**

$$\begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

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Summary

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

r = m = n $\quad \underline{R} = \underline{I}$ ➔ one solution exists (full rank)

r = n < m $\quad \underline{\mathbf{R}} = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}$ ➔ 0 or 1 solution (full column rank)

r = m < n $\quad \underline{\mathbf{R}} = \begin{bmatrix} \mathbf{I} & \mathbf{F} \end{bmatrix}$ ➔ infinitely many solutions (no zero row) (full row rank)

r < m and r < n $\quad \underline{\mathbf{R}} = \begin{bmatrix} \mathbf{I} & \mathbf{F} \\ 0 & 0 \end{bmatrix}$ ➔ no solution or infinite many solutions