EE 422 (2013)
Lecture 2 Static Optimization

- Review
- Optimization with equality constraint
- Optimal Value Function
- Read SYD chapter 3.1- 3.4

Optimization Review

- Goals:
  - able to apply the Lagrange method to solve Optimization with equality constraint
  - Know the application of the Optimum Value Function and Envelop Theorem

Optimization Review

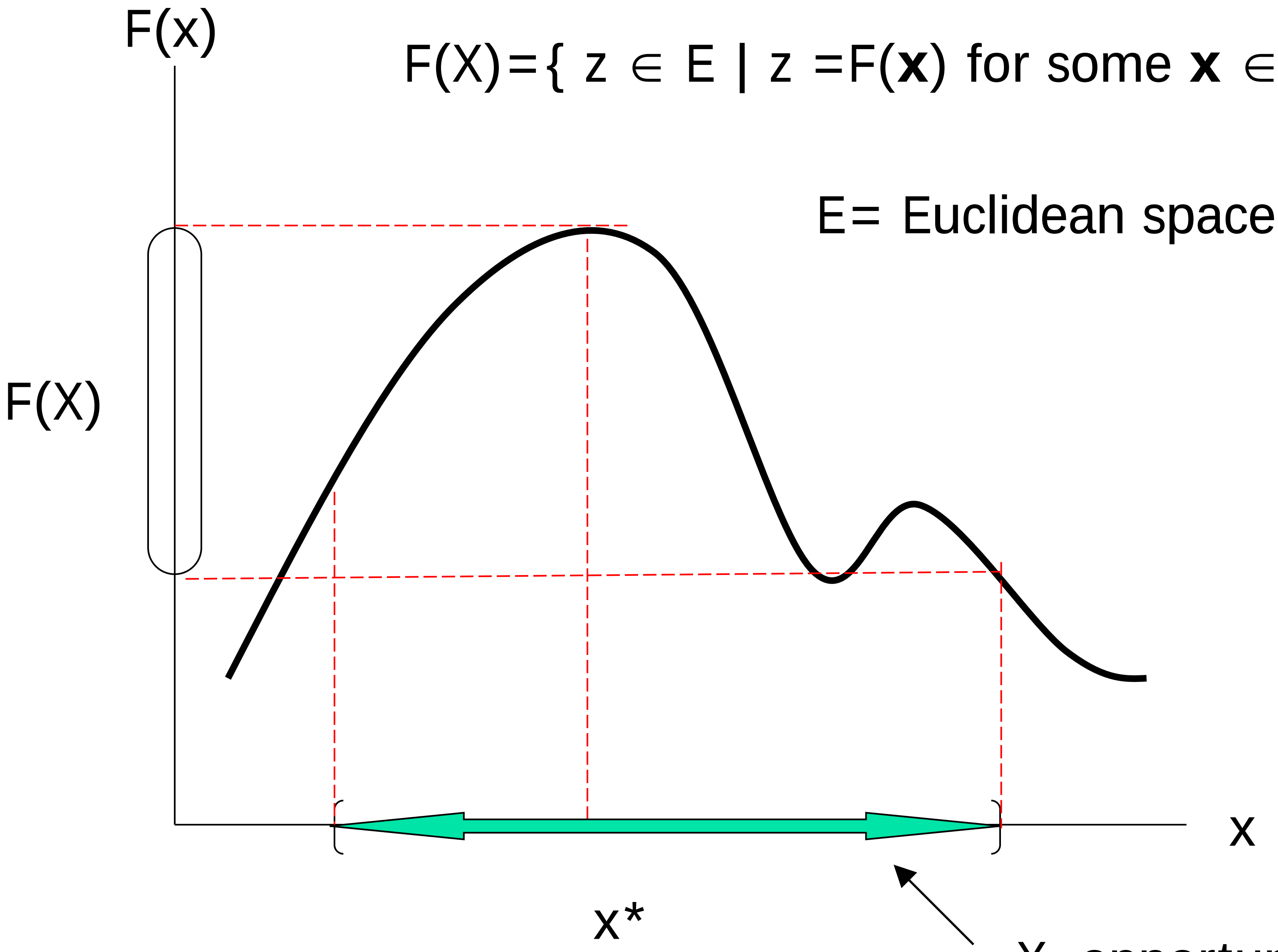
- Type of maxima: global and local
  - Choice vector $\mathbf{x}^*$ is a global maximum (or solution) if it yields a value of the objective function larger than or equal to that obtained by any feasible vector.
  - $\mathbf{x}^* \in X$ and $F(\mathbf{x}^*) \geq F(\mathbf{x})$ for all $\mathbf{x} \in X$
  - Choice vector $\mathbf{x}^*$is a local maximum if it yields a value of $f(\mathbf{x})$ larger than or equal to that obtained by any feasible vector sufficiently close to it.

Weierstrass theorem

- This theorem gives conditions sufficient for the existence of a global maximum.
- If the opportunity set X is compact (since it is a subset of Euclidean n-space) and nonempty and F(**x**) is continuous on X, then F(**x**) has a global maximum, either in the interior or on the boundary of X.
- Note that this theorem is not necessary. Some problem might have a solution even though the opportunity set is not compact.

$$F(X)=\{\, z \in E \mid z = F(\mathbf{x}) \text{ for some } \mathbf{x} \in X \}$$

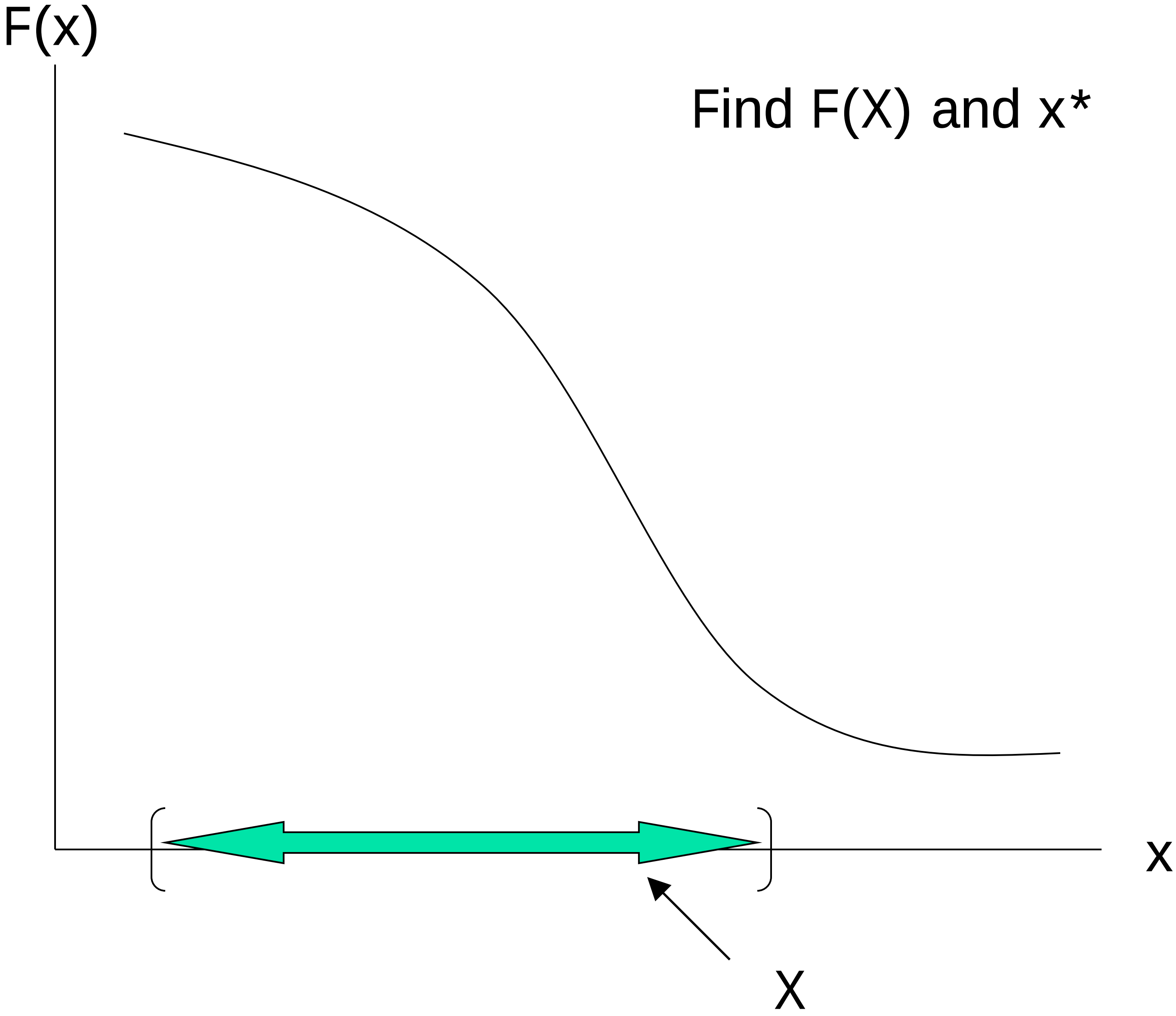
E= Euclidean space



X, opportunity set

Interior solution

Find F(X) and x*



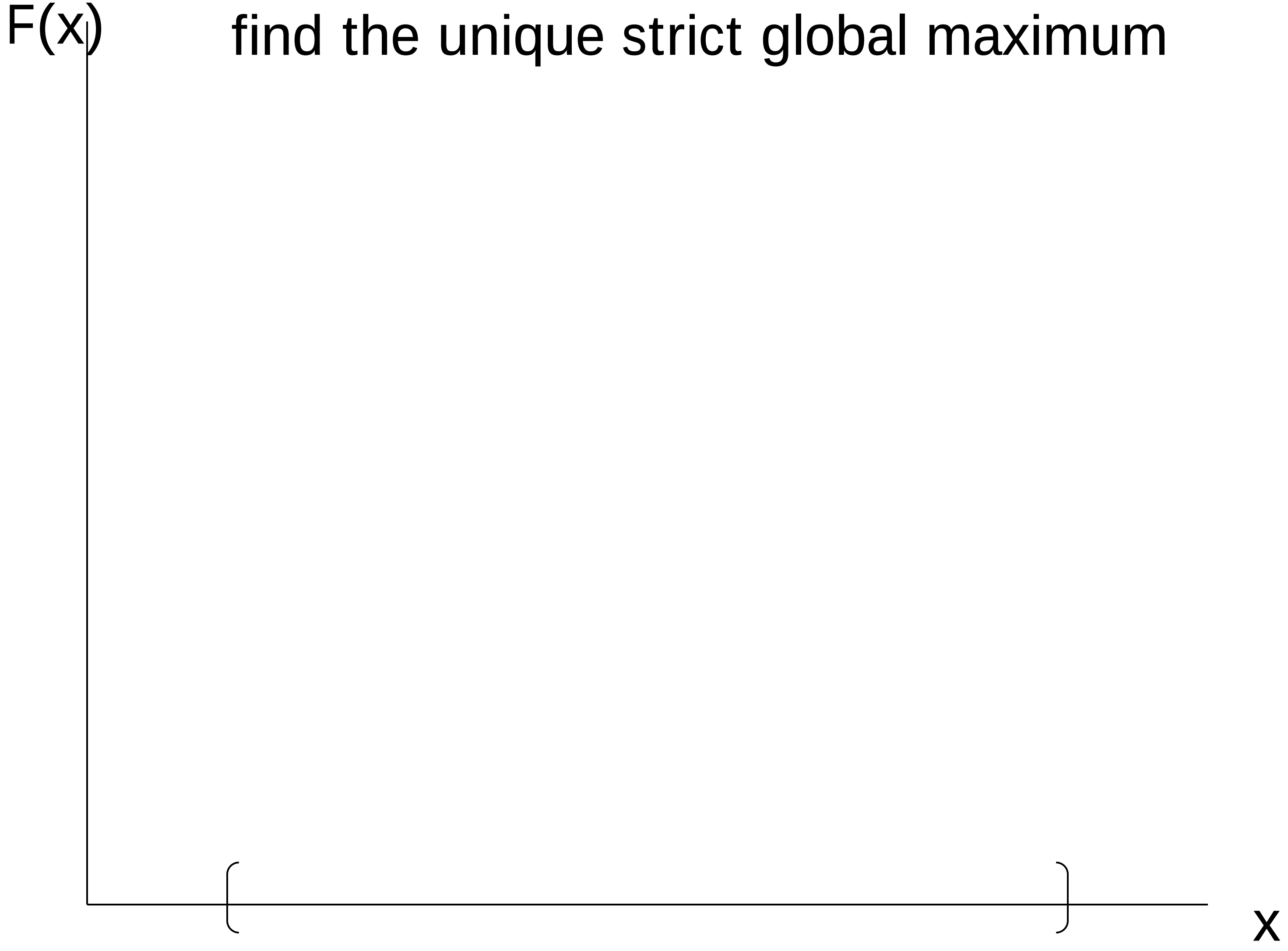
Weierstrass theorem

- Proof. A continuous function defined on a compact set has a compact image, i.e., the set of real numbers. Thus F(X) is compact, and every compact set contains its least upper bound. Thus, if F^* is the least upper bound of F(X), then there is an $\mathbf{x}^* \in X$ satisfying $F(\mathbf{x}^*)=F^*$. Since $F(\mathbf{x}) \le F(\mathbf{x}^*)$ for all $\mathbf{x} \in X$, the point $\mathbf{x}^*$ is a global maximum.

Local-global theorem

- Gives sufficient conditions for a local maximum to be a global maximum.
- “If the opportunity set X is a nonempty compact set that is convex and F(**x**) is a continuous function that is concave over X, then a local maximum is a global maximum. It it is further assumed that F(**x**) is strictly concave, the the solution is also unique, i.e., there is a unique strict global maximum.”

Draw a strictly concave function and find the unique strict global maximum



Constrained Optimization

- 1. Three types of constraints: equality, non-negativity, and inequality.
- 2. Mostly we will focus on equality constraint problem. Consider choosing x_1 and x_2 to maximize

$$f(x_1 , x_2) \text{ subject to } g(x_1 , x_2) = b \ (=0).$$

- 3. x_1 and x_2 are called choice variables.
- 4. f is called the objective function.

Constrained Optimization

- 5. g is called the constraint set, or the feasible set.
- 6. The easiest way to solve this problem is by substitution if we can write x_2 in terms of x_1 from the g function. We then solve the problem in x_1.
- We first show that we can solve this. Then we will show that it is equivalent to the conditions from the Lagrangian method.

Constrained Optimization: by substitution

- Let assume a local solution exists at $\mathbf{x}^*$ and that one of the partial derivatives of the constraint function does not vanish: g_2 or $\partial g(\mathbf{x}^*) / \partial x_2 \neq 0$.
- According to the Implicit function theorem, we can find x_2 as a function of x_1.

Constrained Optimization by substitution

- $dg = g_1\, dx_1 + g_2\, dx_2 = 0$
- $dx_2 / dx_1 = - g_1 / g_2$ and solved for x_2 as a function of x_1 :
- $x_2 = h(x_1)$ where $h' = - g_1 / g_2$ ....(*).
- The problem can then be rewritten as the unconstrained problem in x_1 :
- Max $f\ (x_1, h(x_1))$

Constrained Optimization by substitution

- FONC: $f_1 + f_2 h_1 = 0$ ...(**)
- Use (*) above and get

  $f_1 + f_2 (- g_1 / g_2) = 0$

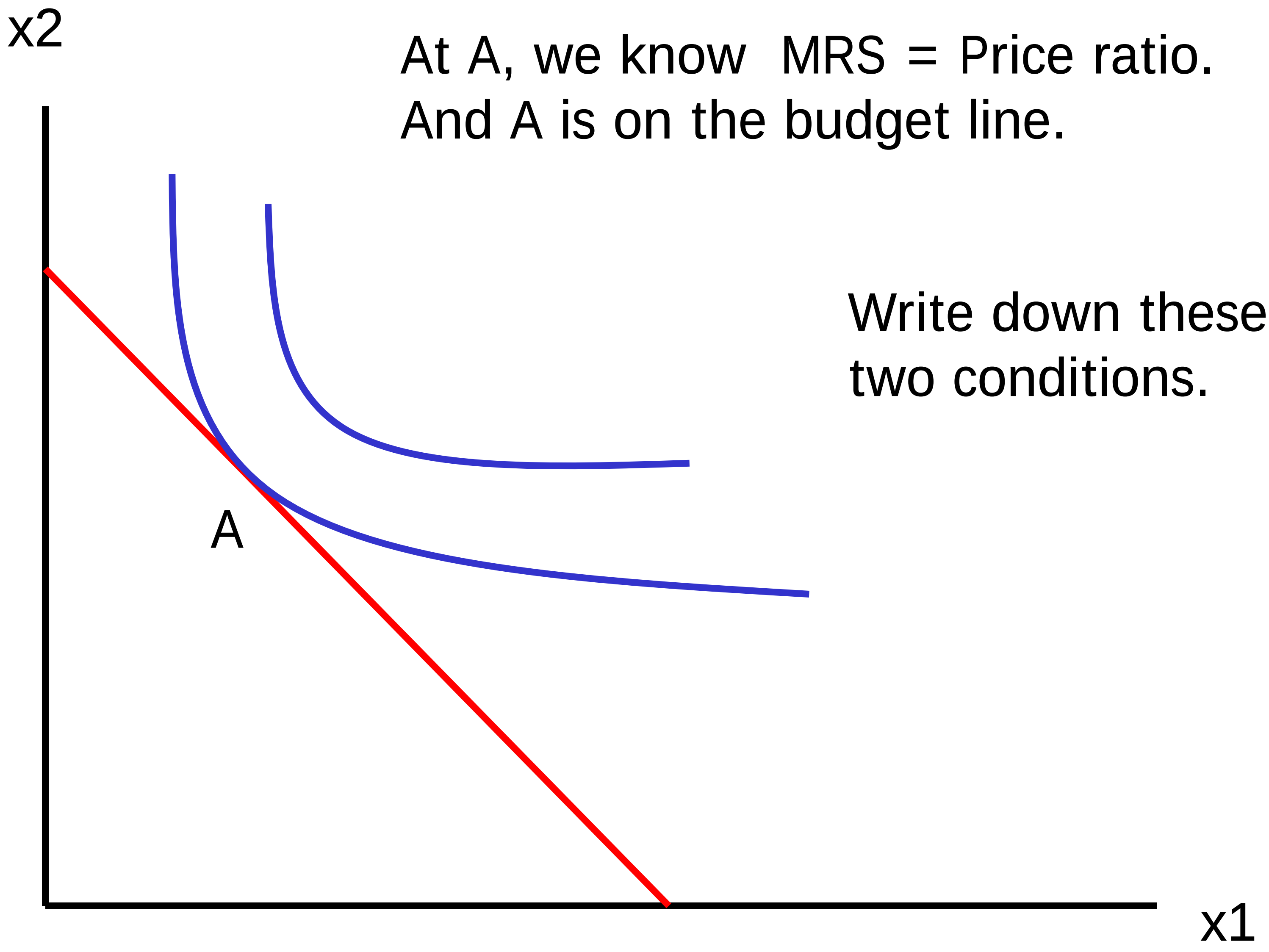
  $f_1 – (f_2 / g_2) g_1 = 0$ .....(1)
- Solve for x^*_1 and $x^*_2=h(x^*_1)$
- Look at (**), it is also true that

  $f_2 – (f_2 / g_2) g_2 = 0$ ......(2)
- Defining λ as (f_2 / g_2), what is (1) / (2)?

Method of Lagrange

- We will start with a standard undergraduate problem and then show that the optimum conditions are exactly the same as we derive from the method of Lagrange.
- Consider the consumer allocation problem: choosing two goods x_1 and x_2 to maximize his satisfaction under his budget constraint.

At A, we know MRS = Price ratio.
And A is on the budget line.



Write down these
two conditions.

Method of Lagrange

- From the picture, we know that at $(x_1{}^*, x_2{}^*)$, $U_1 / U_2 = p_1 / p_2$.
- LHS is MRS. RHS is price ratio. Ignore the minus sign.
- Note that for any level curve $U(x_1, x_2) = 10 \Rightarrow U_1\, dX_1 + , U_2 dX_2 = 0$. So, $dx_2 / dx_1 = - U_1 / U_2$.
- And the budget line equation is $x_2 = (y / p_2) – (p_1 / p_2)\, x_1$.
- Thus, slope of the constraint is $- p_1 / p_2$.

- From the tangency point

$$\frac{U_1}{p_1} = \frac{U_2}{p_2} = \lambda$$

- Let assume λ as a constant and rearrange terms along with the budget condition as

$$U_1 - \lambda p_1 = 0$$

$$U_2 - \lambda p_2 = 0$$

$$p_1 x_1 + p_2 x_2 = I$$

- These are 3 equations for optimal conditions and can be used to solve for 3 unknown x_1 , x_2 , λ.
- What kind of functions that give these conditions?

Method of Lagrange: forming

- It is equivalent to maximize the new L function defined below:

  $L(x_1\ , x_2\ , \lambda) \equiv U\ (x_1\ , x_2)\ +\ \lambda\ [I\ -\ p_1\ x_1\ -\ p_2\ x_2\ ]$

- This is a new function with 3 unknowns but without any constraint.
- λ is called “Lagrange Multiplier” a new unknown needed to be solve along with x_1 and x_2.

Method of Lagrange : forming

- In sum, when we have a following problem :

  max $U(x_1, x_2)$ s.t. $p_1 x_1 + p_2 x_2 = I$, or

  max $U(x_1, x_2)$ s.t. $I - p_1 x_1 - p_2 x_2 = 0$.

- Then we rewrite the problem in terms of L function as :

  $L(x_1, x_2, \lambda) \equiv U(x_1, x_2) + \lambda\,[I - p_1 x_1 - p_2 x_2]$

- FONC: assuming interior solution,

- $L_1 = 0$; => $U_1 - \lambda p_1 = 0$

  $L_2 = 0$; => $U_2 - \lambda p_2 = 0$

  $L_\lambda = 0$; => $p_1 x_1 + p_2 x_2 = I$.

Method of Lagrange

- We can form L(.) in a different way but then the value of λ will be opposite. If we don't care about its value, we are just fine.
- In economics, however, the sensitivity of the objective function to changes in constraint constant has a beautiful meaning. It is the price of resource constraint. So, we will make the new term in [..] positive, and λ will have a positive value.

Ex. Max x_1x_2 st. $x_1 + 4x_2 = 16$

First, write down the Lagrangian function as

$L(x_1, x_2, \lambda) = \ldots..$

FOC :

$L_1 = 0$; => ...

$L_2 = 0$; => ...

$L_\lambda = 0$; => ...

$x_1^* = \ldots\ldots$

$x_2^* = \ldots\ldots$

Maximized value of the obj function = 16.

The shadow price of the constraint = ……

Method of Lagrange : second order necessary conditions

The optimum values we find may not be the maximum, it could be minimum. In other words, those are only necessary conditions for maximum.

If **x*** is maximum, then f'(**x***)=0.

We need to check a second-order condition to make sure that we get the answer.

Two methods to state the 2nd necessary conditions:

Method of Lagrange : second order necessary conditions

- Remind that in unconstrained problem, we check the second order partial derivatives, which is equivalent to check the curvature of the objective function. Assuming FONC holds,
- SOSC for one variable: f" (x) < 0 . [some can occur at f" (x) = 0]. Allow both we have
- SONC for one variable: f" (x) $\leq$ 0, or
- SONC for two variables: H(**x**) is negative semidefinite

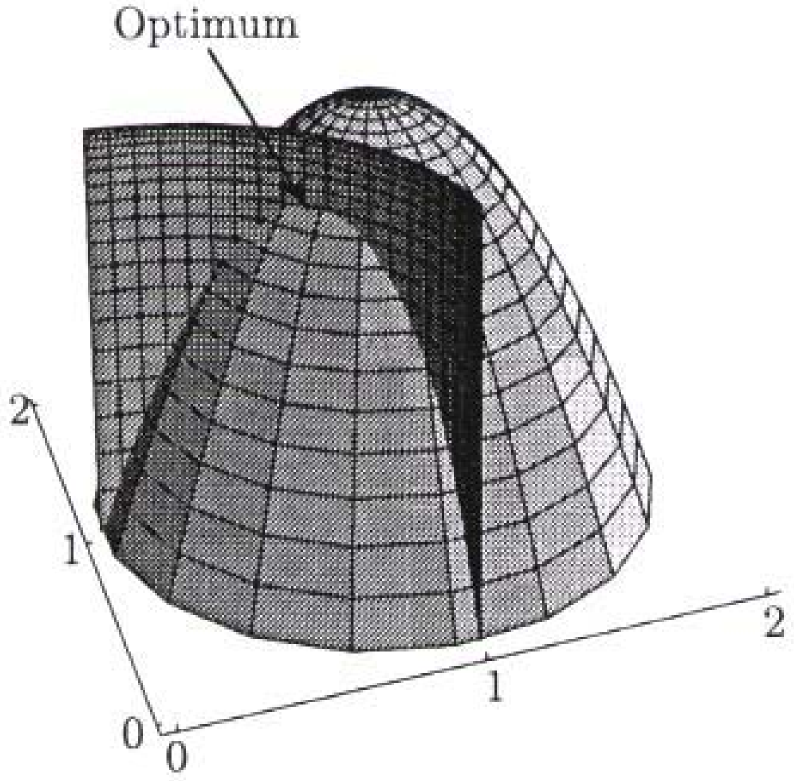
Method of Lagrange : second order necessary conditions

- For constrained problem, SONC requires that H(x) of the Lagrangian evaluated at the local maximum (**x***, λ*) must be negative semidefinite when subject to the constraint.
- Intuitively, we check the curvature of the objective function at the optimum along the constraint. For a maximum, this second-order differential must be decreasing along the constraint.
- We need H be negative semidefinite for any change in a direction tangent to the constraint surface.

Method of Lagrange : second order <u>sufficient</u> conditions

- If H is negative <u>definite</u> subject to the constraint, we call that a regular maximum.
- This is equivalent to say that determinant of bordered Hessian > 0.
- This is SOSC, since we could have a solution where determinant of bordered Hessian = 0.
- Again, our knowledge about the objective function could be sufficient to identify the maximum.
- For constrained problem, strictly quasiconcave obj function gives this second-order sufficient condition.
- This obj function yields strictly convex level curves.

Method of Lagrange:



The optimum is the highest point that is common to the objective surface and the constraint

Method of Lagrange: second order conditions

- The curvature of the obj fn. along the constraint can be inferred from the sign of the det of the bordered Hessian of L.
- For a problem of max $f(\mathbf{x})$ st. $g(\mathbf{x}) = 0$
- $L(\mathbf{x}, \lambda) = f(x) - \lambda\, g(\mathbf{x})$
- FONC: $L_\lambda = 0 \quad => \; \ldots$

  $L_1 = 0 \quad => \; \ldots$

  $L_2 = 0 \quad => \; \ldots$

Method of Lagrange: second order conditions

- Hessian matrix of L is

$$|\bar{\mathbf{H}}| = \begin{vmatrix} L_{\lambda\lambda} & L_{\lambda 1} & L_{\lambda 2} \\ L_{1\lambda} & L_{11} & L_{12} \\ L_{2\lambda} & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

- Determinant of this “bordered Hessian” is $-[L_{11}(g_2)^2 - 2L_{12}g_1g_2 + L_{22}(g_1)^2]$

Method of Lagrange : second order conditions

- Alternative SOC states that the bordered Hessian matrix (size 2+1=3) has a <u>positive determinant in case of a regular maximum</u>.
- When we have n variables and 1 constraint. The bordered H is (n+1) by (n+1) matrix. In this case, we need to look at the determinants of various submatrices of the bordered H.

Method of Lagrange : second order conditions

- The SOC requires that the last n-m leading principal minors of the bordered H must alternate in sign, starting with $(-1)^{m+1.}$
- For example, n=2, m=1. Start with $(-1)^2$ or positive.

$$\begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0.$$

Method of Lagrange : second order conditions

- For example, n=3, m=1. We need last two leading principal minors to alternate in sign, starting with $(-1)^2$ or positive.
- That is $|\overline{H}_2| > 0$, and $|\overline{H}_3| < 0$.

$$\begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0;\ and \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix} < 0.$$

Method of Lagrange : second order conditions

- Ex 1. Max x_1x_2 st. $x_1 + (1+r)^{-1} x_2 = B$.
- $L = x_1x_2 + \lambda [B - x_1 - (1+r)^{-1} x_2]$
- FONC:
- $x_1^{*} = B/2$, $x_2^{*} = B(1+r)/2$, $\lambda^{*} = B(1+r)/2$
- SOC requires that the last n-m leading principal minors of the bordered H must alternate in sign, starting with $(-1)^{m+1}$.
- So, we need the last leading principal minors to be positive.

Method of Lagrange : second order conditions

$$\begin{vmatrix} 0 & -1 & -(1+r)^{-1} \\ -1 & 0 & 1 \\ -(1+r)^{-1} & 1 & 0 \end{vmatrix} = 2/(1+r) > 0.$$

So, the second-order sufficient condition holds for a maximum.

Method of Lagrange : second order conditions

- Ex 2. Max U(x, y) st. $p_x\, x + p_y y = B$.
- $L = U(x, y) + \lambda\,[B - p_x x - p_y y\,]$
- SOC: Need Bordered Hessian det > 0
- Bordered H = $\begin{pmatrix} 0 & p_x & p_y \\ p_x & U_{xx} & U_{xy} \\ p_y & U_{yx} & U_{yy} \end{pmatrix}$
- At the stationary value, det of bordered H=

$$2p_x p_y U_{xy} - p^2{}_y U_{xx} - p^2{}_x U_{yy} > 0$$

Method of Lagrange : second order conditions

$$ex.\max u(x,y) = x^a y^b \quad (x,y > 0; 0 < a,b < 1)$$

$$u_x = ax^{a-1}y^b; \quad u_{xx} = a(a-1)x^{a-2}y^b$$

$$u_y = bx^a y^{b-1}; \quad u_{yy} = b(b-1)x^a y^{b-2}$$

$$\bar{H} = \begin{bmatrix} 0 & p_x & p_y \\ p_x & u_{xx} & u_{xy} \\ p_y & u_{yx} & u_{yy} \end{bmatrix}, we\,need \left|\bar{H}\right| at\,optimum > 0.$$

$$Show\,that\,u\,is\,strictly\,quasiconcave, that\,is$$

$$\frac{d^2y}{dx^2} = \frac{\left|\bar{H}\right|}{u_y p_y^2} > 0.$$

Method of Lagrange: caution

- This method works if both g_1 and g_2 not equal to 0 at x*. Or the gradient of the constraint function at x* is not the zero vector.
- This is a restriction on the constraint set and it is called “constraints qualification” or CQ.
- If the constraint is linear, CQ is satisfied. (check).
- How about max $(1+x_1)(1-x_2)$

  subject to $g(x_1, x_2) = x_1^3 - x_2^3 = 0$

Method of Lagrange: caution

- From constraint, $x_1 = x_2$. Sub this in the objective fn, we get $(1+x_1)(1 - x_1)$. The solution is where both are 0.
- Now, let form

  $$L = (1+x_1)(1-x_2) + \lambda\,[0 - x_1^3 + x_2^3]\ .$$
- Show that $(0,0,\lambda)$ is not a critical point of L for any λ. At this $x_1 = x_2 = 0$, $g_1 = g_2 = 0$, thus violating CQ. Thus, the L method fails.

Method of Lagrange: caution

- In practice, we find critical points of the constraint function g(.), that is solutions of g_1 and g_2 =0. If none of those lie in the constraint set, we can use the Lagrange method.
- If the constraint set does contain a critical point of the constraint function, then we include this point among our candidates for a solution to our original problem, along with the critical points of the Lagrangian.

Method of Lagrange: caution

- When we have more than one equality constraints, say m. We will need the number of constraints should not exceed the number of variables.
- Generalization of CQ for one function of 2 variables [$(g_1 \;\; g_2) \neq (0, 0)$] to m functions involves the Jacobian matrix.
- CQ needs the Jacobian matrix of constraints to be full rank at x*. This is a regularity condition.

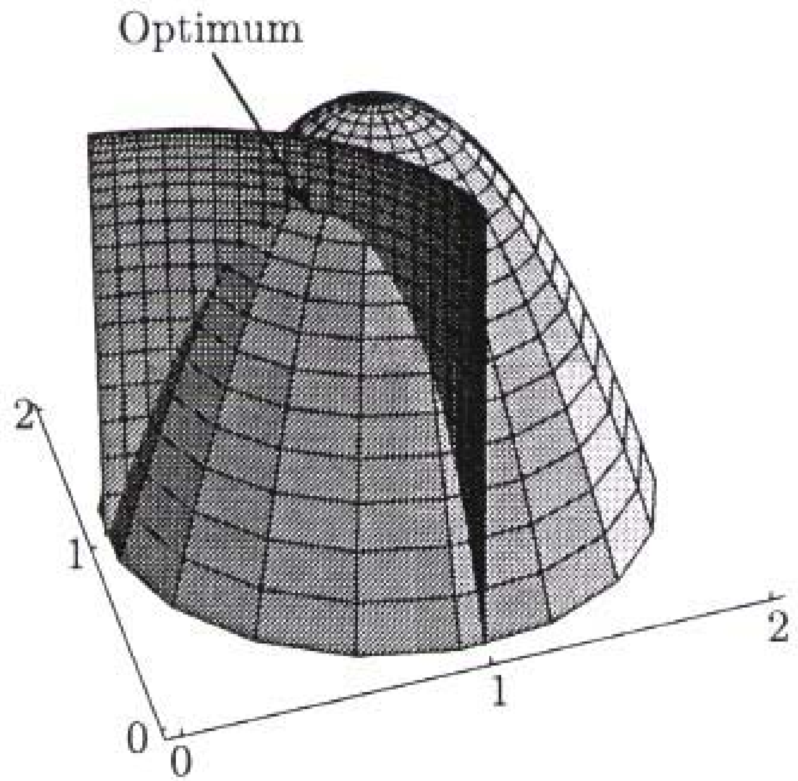
$$\begin{pmatrix} \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g^m}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_n} \end{pmatrix} \begin{matrix} \longrightarrow & \nabla g^1(\mathbf{x}^*) \\ & \\ \longrightarrow & \nabla g^m(\mathbf{x}^*) \end{matrix}$$

$$\lambda_1 \nabla g^1(\mathbf{x}^*) + \ldots + \lambda_m \nabla g^m(\mathbf{x}^*) = \nabla f(\mathbf{x}^*)$$

Method of Lagrange: caution

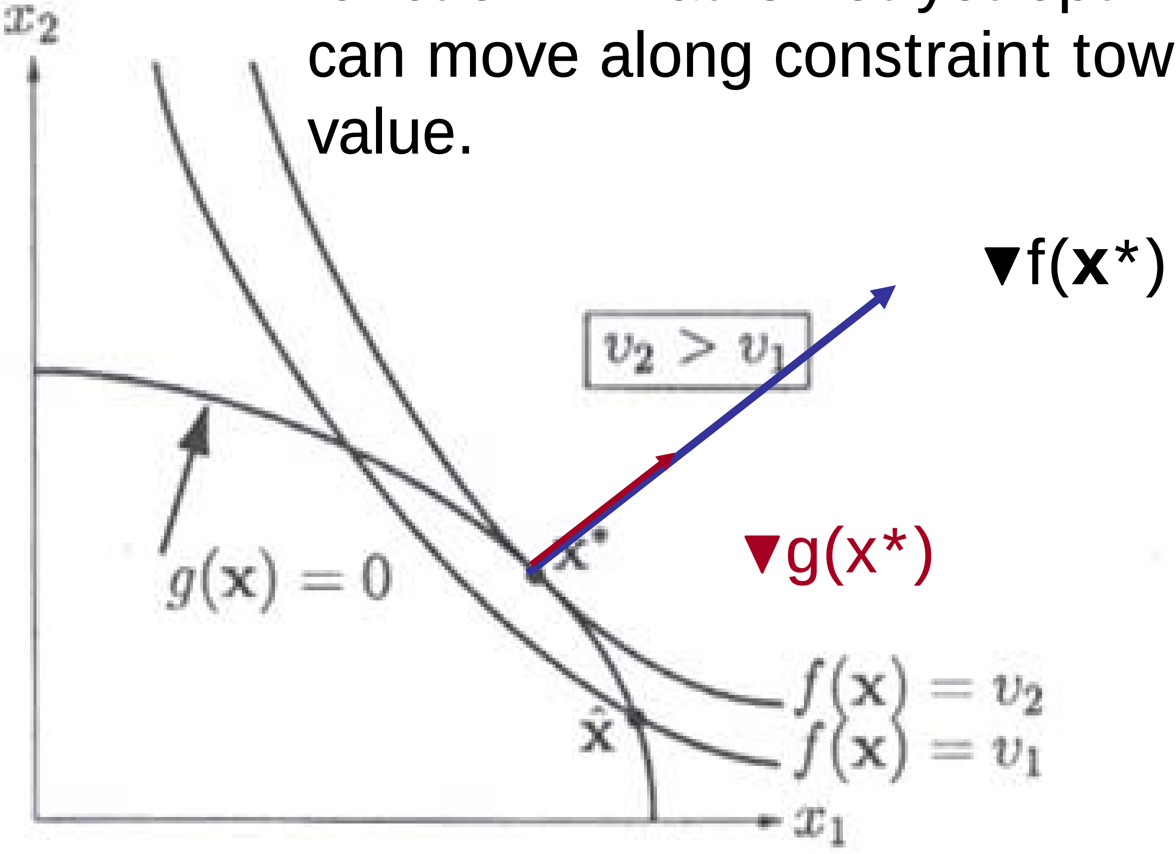
- This implies that the constraint set has a well-defined n-m dimensional tangent plane everywhere.
- Or gradients of constraints are linearly independent. (more later) That is, no gradient vector of constraints can be expressed as a linear combination of the others.

Method of Lagrange:



The optimum is the highest point that is common to the objective surface and the constraint

Idea: a small movement along constraints cannot improve the value of the objective function. Xhat is not yet optimum since we can move along constraint toward higher f value.



$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \lambda \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

and

$$\frac{f_1}{f_2} = \frac{g_1}{g_2}$$

Tangency between the constraint and the objective function

Method of Lagrange:

- The gradient vectors of f and g at x* point in the same direction as in the picture <u>or</u> in opposite direction (not shown).
- Remind that $U_1 - \lambda p_1 = 0$

  $U_2 - \lambda p_2 = 0$
- Or

$$\begin{pmatrix} \dfrac{U_1}{U_2} \end{pmatrix} = \begin{pmatrix} \dfrac{p_1}{p_2} \end{pmatrix} at\, \mathbf{x}^*$$

Summary of Lagrangean Method

- 1. Forming the L function and make sure that the term after lambda is positive.

$$L(x_1 , x_2 , \lambda) \equiv f(x_1 , x_2) - \lambda\, g(x_1 , x_2)$$

$$\text{or } = f(x_1 , x_2) - \lambda\, (g(x_1 , x_2)-c)$$

$$\text{or } = f(x_1 , x_2) +\lambda\, (c-g(x_1 , x_2))$$

- 2. Set the partial derivative of the L function with respect to x_1, x_2, and λ equal to zero. Here, we get three equations.
- 3. To solve three equations for three unknowns, first get rid of λ (using the first and second equations), and write x_1 in terms of x_2, and then using this fact in the constraints or the third equation.

Summary of Lagrangean Method

- 4. Your optimal value of x_1, x_2 and λ must depend only on parameters.
- 5. When plugging these optimal values in the objective functions, we call the new function, that is, a maximum value function for a maximization problem and a minimum value function for a minimization problem.
- 6. The optimal value function is a function of parameters.

Summary of Lagrangean Method

- 7. Second order conditions is to check the bordered Hessian. Graphically, we look at the curvature of the objective function along the constraint.
  - 7.1 The bordered Hessian of the Lagrangean function can be written as

$$\bar{H} = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{bmatrix}$$

$$\max f(\mathbf{x})\ st.\ g(\mathbf{x})=0$$

$$L(\lambda,x_1,x_2)\equiv f(x_1,x_2)-\lambda g(x_1,x_2)$$

$$\frac{\partial L}{\partial x_1}=f_1-\lambda g_1=0;\Rightarrow\frac{\partial^2 L}{\partial x_1\partial\lambda}=-g_1.$$

$$\frac{\partial L}{\partial\lambda}=g(\mathbf{x})=0;\Rightarrow\frac{\partial^2 L}{\partial\lambda^2}=0.$$

$$\bar{H}=\mathbf{D}^2L(\lambda,x_1,x_2)$$

$$=\begin{pmatrix}\frac{\partial^2 L}{\partial\lambda^2} & \frac{\partial^2 L}{\partial\lambda\partial x_1} & \frac{\partial^2 L}{\partial\lambda\partial x_2}\\ \frac{\partial^2 L}{\partial x_1\partial\lambda_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1\partial x_2}\\ \frac{\partial^2 L}{\partial x_2\partial\lambda_2} & \frac{\partial^2 L}{\partial x_2\partial x_1} & \frac{\partial^2 L}{\partial x_2^2}\end{pmatrix}=\begin{pmatrix}0 & -g_1 & -g_2\\ -g_1 & L_{11} & L_{12}\\ -g_2 & L_{21} & L_{22}\end{pmatrix}$$

Summary of Lagrangean Method

- 8. Quick check of SOC for a constrained maximization problem with two variables is to verify that the determinant of bordered Hessian > 0 evaluated at (x_1^*, x_2^*, λ^*).
- Notice that if we multiply the first row and first column by –1, the sign of determinant of bordered H will not change. Thus we can work with

$$\bar{H} = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{bmatrix}$$

Summary of Lagrangean Method

- For example. Let start with a matrix that is negative semidefinite subject to one constraint.
- Its bordered Hessian is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$
- Two leading principal minors are

$$\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 < 0 \quad \text{and} \quad |H| = 2 > 0$$

Summary of Lagrangean Method

- For n=3 or three choice variables, the bordered Hessian will be an 4x4 matrix. For a regular local maximum, we need the leading principal minor of the bordered Hessian alternates in sign, starting with >0.

Summary of Lagrangean Method

- As for Min problem, SOC needs the leading principal minor of bordered Hessian < 0 for all minors.
- For example $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
- Two leading principal minors are -1 and –2

Summary of Lagrangean Method

- How to remember this. Remind that
- A positive definite matrix is the identity Matrix. See that the principal minors of identity matrix are all positive.
- A negative definite matrix is the negative of the identity matrix. Now check that the principal minors of this matrix must alternate in sign.

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} > 0; \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} < 0.$$

Summary of Lagrangean Method

- 9. The meaning of the Lagrange multiplier is known by using the envelope theorem. It tells how how sensitive the optimal value of the obj function is to changes in the constraint constants. In utility maximization problem, it is a marginal utility of income. In cost minimization problem, it is a marginal cost.
- 10. When the Lagrange multiplier has a value (not equal zero), then this implies a shadow price of resources. Equivalently, it is to say that the constraint is binding, or resource is scarce and be exhausted.
- 11. When it is zero at the solution, then small changes in the constraint constant would not affect the optimal value of the obj function. Thus, its shadow price is zero.

Summary of Lagrangean Method

- The method can be extended to n variables and m constraints
- max $F(\mathbf{x})$ subject to $\mathbf{g(x)} = \mathbf{b}$. Or
- max $F(x_1, \ldots, x_n)$ subject to

$$\begin{aligned} g^1(x_1, \ldots, x_n) &= b_1 \\ g^2(x_1, \ldots, x_n) &= b_2 \\ \vdots \quad & \quad \vdots \\ g^m(x_1, \ldots, x_n) &= b_m \ . \end{aligned}$$

- $L(\mathbf{x}, \boldsymbol{\lambda}) = F(\mathbf{x}) + \boldsymbol{\lambda}\,(\mathbf{b} - \mathbf{g(x)})$
- Solving the m+n equations yields solutions for the m+n unknowns. x^* is a local solution where $L(\mathbf{x^*}, \boldsymbol{\lambda^*}) = F(\mathbf{x^*})$.

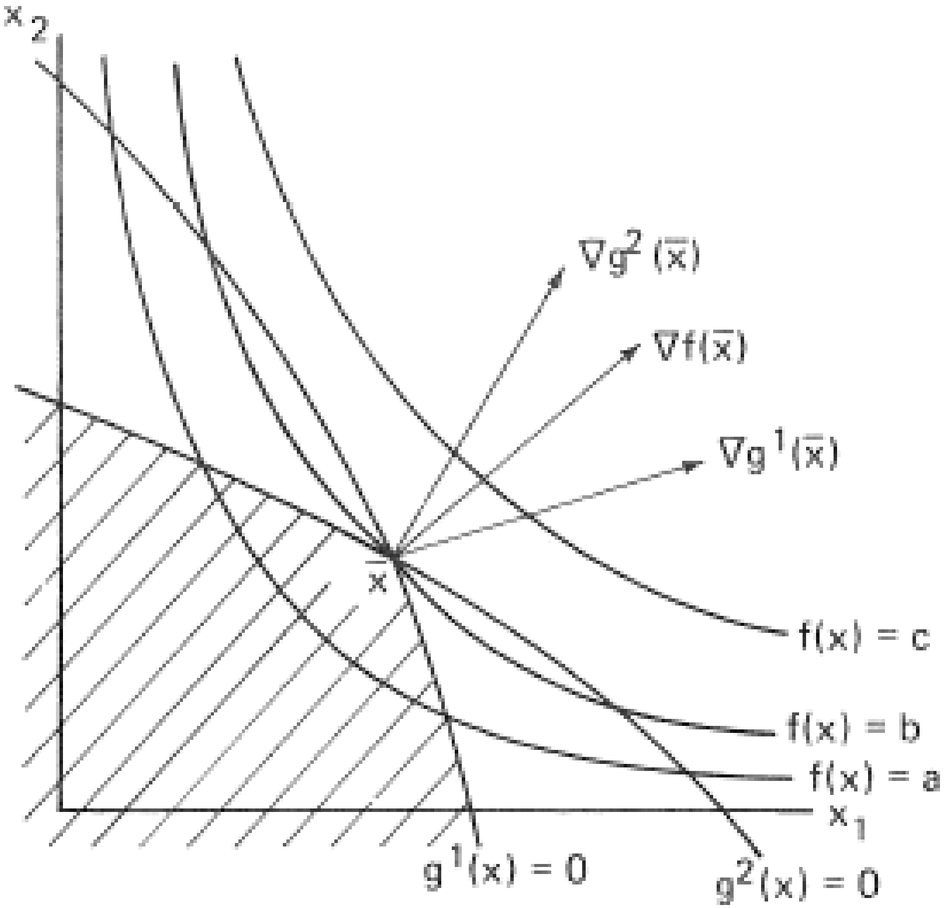
Summary of Lagrangean Method

- $L(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda\,(\mathbf{b} - \mathbf{g}(\mathbf{x}))$
- $L = F(\mathbf{x}) + \lambda_1(b_1 - g^1(\mathbf{x})) + \ldots + \lambda_m(b_m - g^m(\mathbf{x}))$
- $L = F(\mathbf{x}) - \lambda_1 g^1(\mathbf{x}) - \ldots - \lambda_m g^m(\mathbf{x})$
- FOC : $F_i - \sum_j \lambda_j g_j^i(\mathbf{x}) = 0 \quad , \; i = 1, \ldots, n.$
- This can be expressed as

$$\nabla F(\mathbf{x}^*) = \lambda_1 \nabla g^1(\mathbf{x}^*) + \ldots + \lambda_m \nabla g^m(\mathbf{x}^*)$$

where $\nabla g^1(\mathbf{x}^*)$ is the row vector of Jacobian of $\mathbf{g}(\mathbf{x})$. Thus, this is equivalent to the condition that the gradients are linearly independent. (See SYD p. 118)

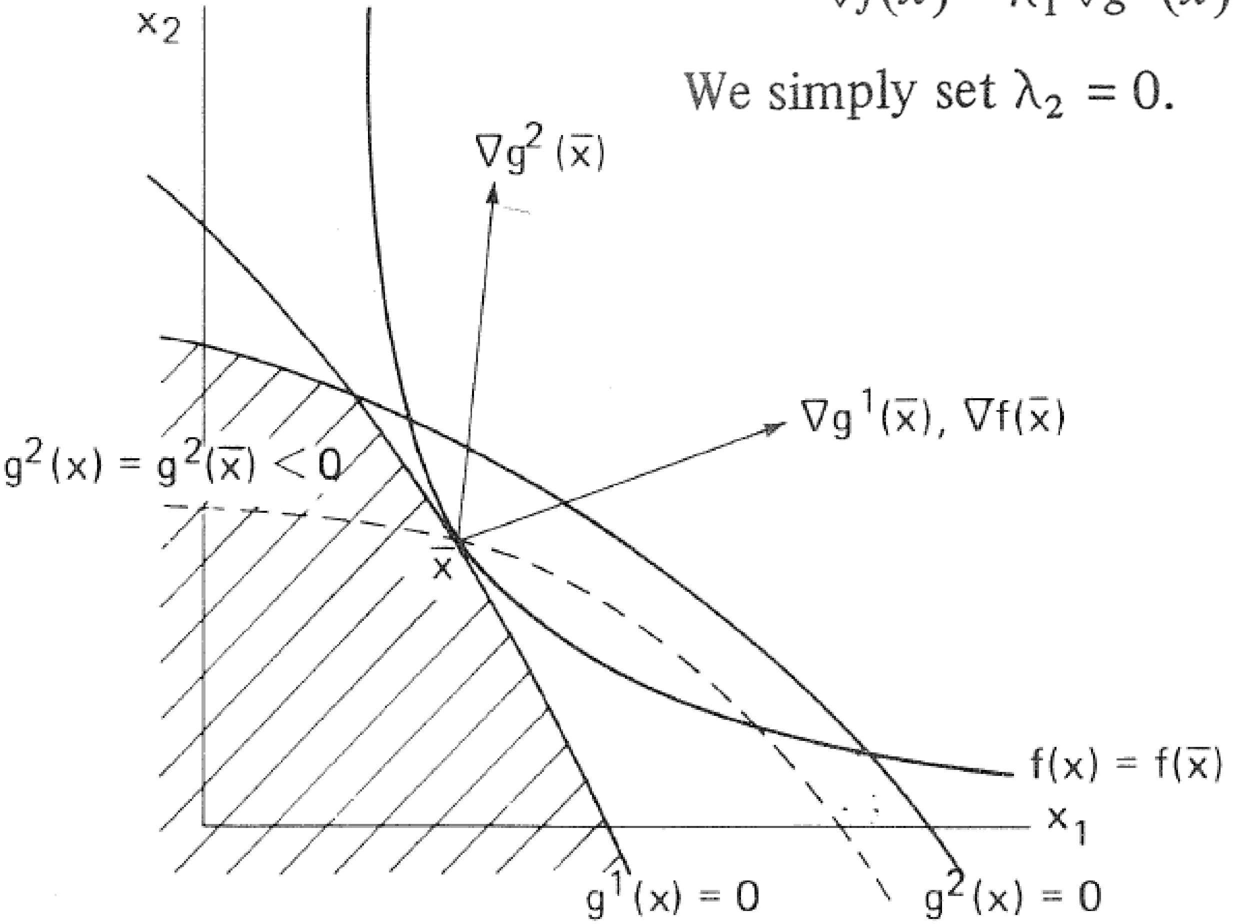
$$\nabla f(\bar{x}) = \lambda_1 \nabla g^1(\bar{x}) + \lambda_2 \nabla g^2(\bar{x}).$$

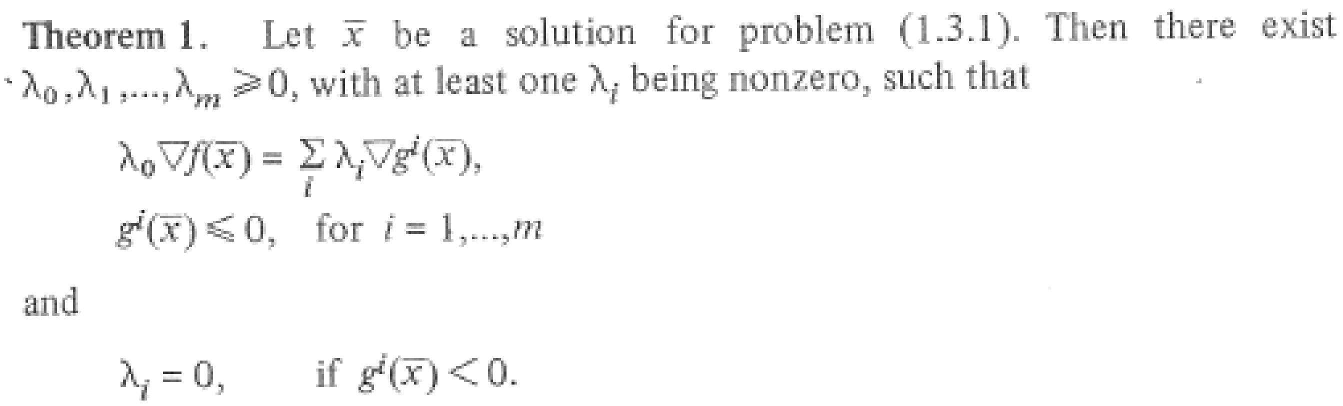


shows only one of the constraints as binding at the solution $\bar{x}$. The existence of the second constraint has no influence on the problem solution. Once more, we see that there exists $\lambda_1, \lambda_2 \geqslant 0$ such that

$$\nabla f(\bar{x}) = \lambda_1 \nabla g^1(\bar{x}) + \lambda_2 \nabla g^2(\bar{x}).$$

We simply set $\lambda_2 = 0$.





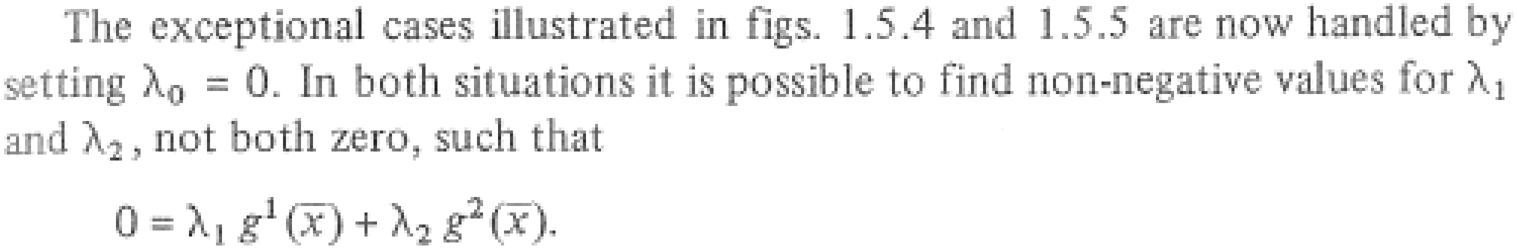
Theorem 1. Let $\bar{x}$ be a solution for problem (1.3.1). Then there exist $\lambda_0, \lambda_1, \ldots, \lambda_m \geqslant 0$, with at least one λ_i being nonzero, such that

$$\lambda_0 \nabla f(\bar{x}) = \sum_i \lambda_i \nabla g^i(\bar{x}),$$

$$g^i(\bar{x}) \leqslant 0, \quad \text{for } i = 1, \ldots, m$$

and

$$\lambda_i = 0, \qquad \text{if } g^i(\bar{x}) < 0.$$

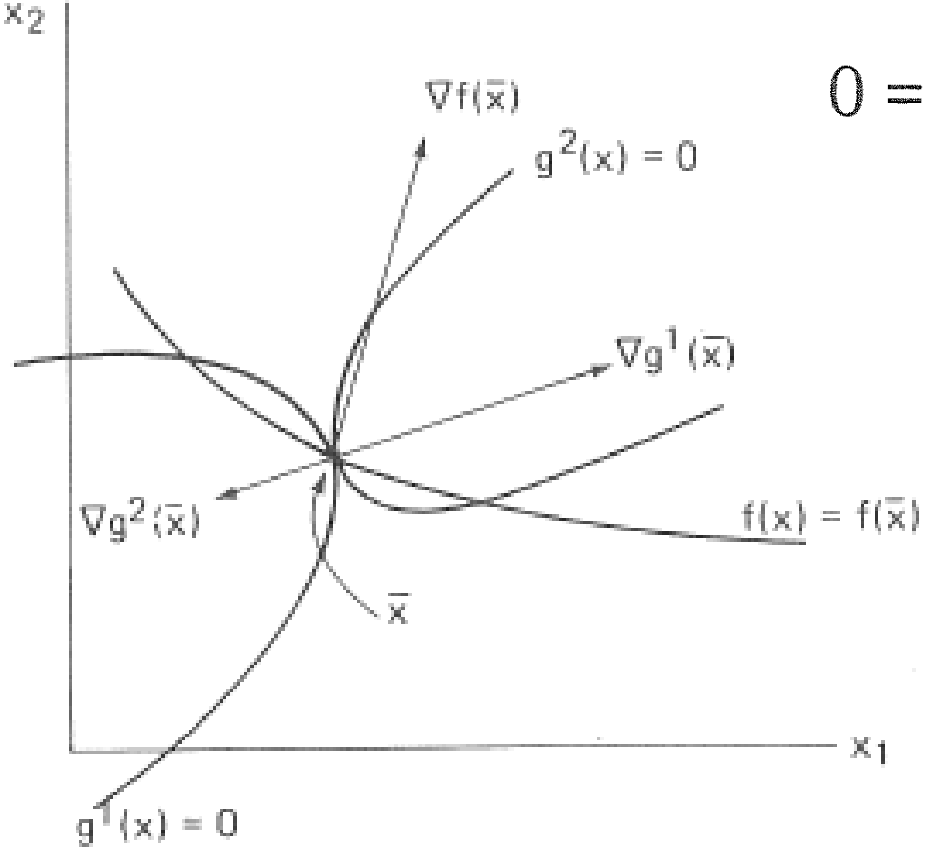


The exceptional cases illustrated in figs. 1.5.4 and 1.5.5 are now handled by setting $\lambda_0 = 0$. In both situations it is possible to find non-negative values for λ_1 and λ_2, not both zero, such that

$$0 = \lambda_1 g^1(\bar{x}) + \lambda_2 g^2(\bar{x}).$$

See next two pictures for the exceptional cases

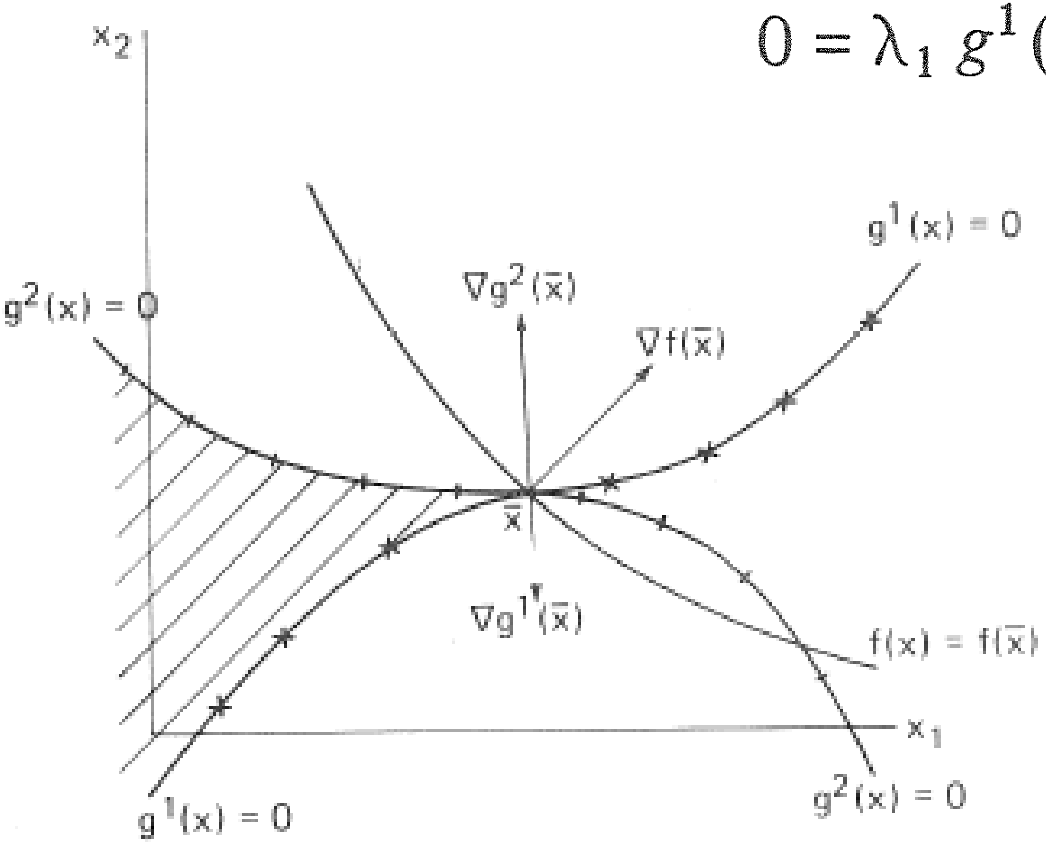
In this case, xbar is the only point in the feasible set and the only solution. And we cannot find non-negative lamda's such that $\nabla f(\bar{x}) = \lambda_1 \nabla g^1(\bar{x}) + \lambda_2 \nabla g^2(\bar{x})$.



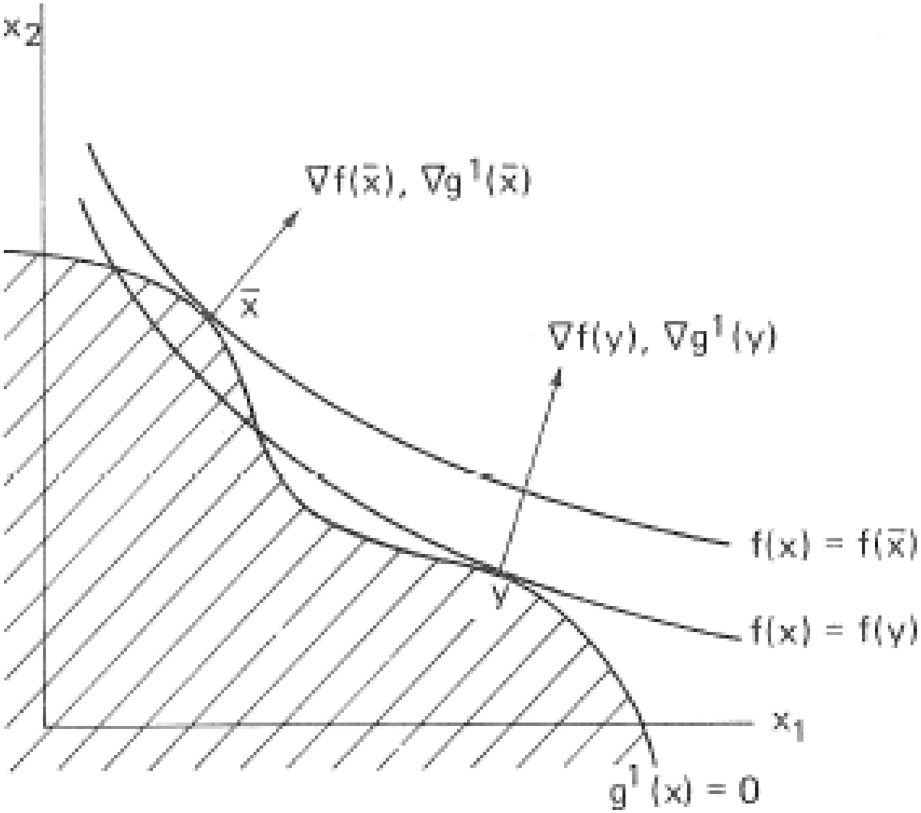
$$0 = \lambda_1 \, g^1(\bar{x}) + \lambda_2 \, g^2(\bar{x}).$$

In this case, xbar is the only point in the feasible set and the only solution. And we cannot find non-negative lamda's such that $\nabla f(\bar{x}) = \lambda_1 \nabla g^1(\bar{x}) + \lambda_2 \nabla g^2(\bar{x})$.

$$0 = \lambda_1 \, g^1(\bar{x}) + \lambda_2 \, g^2(\bar{x}).$$



Theorem 1 is necessary but not yet sufficient



2. Value Function

Value function

- Consider Max f(**x**, **a**)   s.t. g(**x**, **a**) = 0
- We call "**a**" parameters
- Solution we get must be expressed in terms of **a**. That is, we have **x*=x**(**a**)
- If we put x(a) in our objective function, we get M(**a**) $\equiv$ f(**x**(**a**), **a**).
- We call M(**a**) as **a maximum value function.**

Value function

EX. Max $U(x_1, x_2) = x_1 x_2$ s.t. $x_1 + 4x_2 = a$

$L(x_1, x_2, \lambda) \equiv x_1 x_2 + \lambda [a - x_1 - 4x_2]$

FONC:

$L_1 = 0$; => $x_2 - \lambda = 0$ ....(1)

$L_2 = 0$; => $x_1 - \lambda 4 = 0$ ......(2)

$L_\lambda = 0$; => $a - x_1 - 4x_2 = 0$ .....(3)

Using (1) and (2), we get $x_1 = 4x_2$

Sub $x_1 = 4x_2$ in (3), we get $a - 4x_2 - 4x_2 = 0$

Thus $x_2^* = a/8$; $x_1^* = a/2$; $\lambda^* = a/8$

And $U(a) = x_1^* x_2^* = a^2/16$

Value function

- The value function gives the value when choice variables are chosen to maximize f subject to the constraints.
- M(**a**) tells us how the maximum value of function will change as the value of **a** changes.
- If we already knew M(**a**), we can do the calculation. From previous example,

  $$\partial U(a) / \partial a = a/8$$

- Note that it happens to be equal to λ *

Value function

- If we want to know how the solutions to the maximization problem vary with parameters,
- We can redo the problem with a new parameter, or
- We can apply the envelope theorem
- The Envelop theorem gives us a formula for the derivative of the value function w.r.t a parameter in the optimization problem.

Value function

- Formula for the Envelope theorem:

  Total effect on the optimized value of the objective function when a parameter changes can be calculated by

$$\frac{dM(a)}{da_i} = \left.\frac{\partial L}{\partial a_i}\right|_{x(a),\lambda(a)}$$

- measure changes around the optimal values.

$$M(a) \equiv f(x_1(a), x_2(a), a)$$

$$\frac{dM(a)}{da_i} = \left.\frac{\partial L(x,a)}{\partial a_i}\right|_{x(a)} = \left.\frac{\partial f(x,a)}{\partial a_i}\right|_{x(a)} - \lambda \left.\frac{\partial g(x,a)}{\partial a_i}\right|_{x(a)}$$

$$Proof.\left(assuming\ differetiability\right)$$

$$\frac{dM(a)}{da_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial a_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial a_i} + \frac{\partial f}{\partial a_i}.$$

$$= \lambda \underbrace{\left[\frac{\partial g}{\partial x_1}\frac{\partial x_1}{\partial a_i} + \frac{\partial g}{\partial x_2}\frac{\partial x_2}{\partial a_i}\right]} + \frac{\partial f}{\partial a_i} \qquad use\, FOC$$

$$= \lambda \left[-\frac{\partial g}{\partial a_i}\right] + \frac{\partial f}{\partial a_i} \qquad use\, diff\ g(x_1(a), x_2(a), a) = o$$

Value function

- We start the Envelope theorem with assuming that M(a) and x(a) are continuous in parameters a and also differentiable.
- The continuity properties comes from the Theorem of Maximum which states that : if the objective function and the constraint are continuous in parameters, and if the domain is a compact set, then M(a) and x(a) are continuous functions of parameters.

Value function

- Idea: the value function changes because of two effects.
- First, the indirect effect, x(a) changes so that f(a) changes.
- Secondly, the direct effect of change in a.
- Since x(a) is optimal, so the indirect effect will have a negligible influence on f.

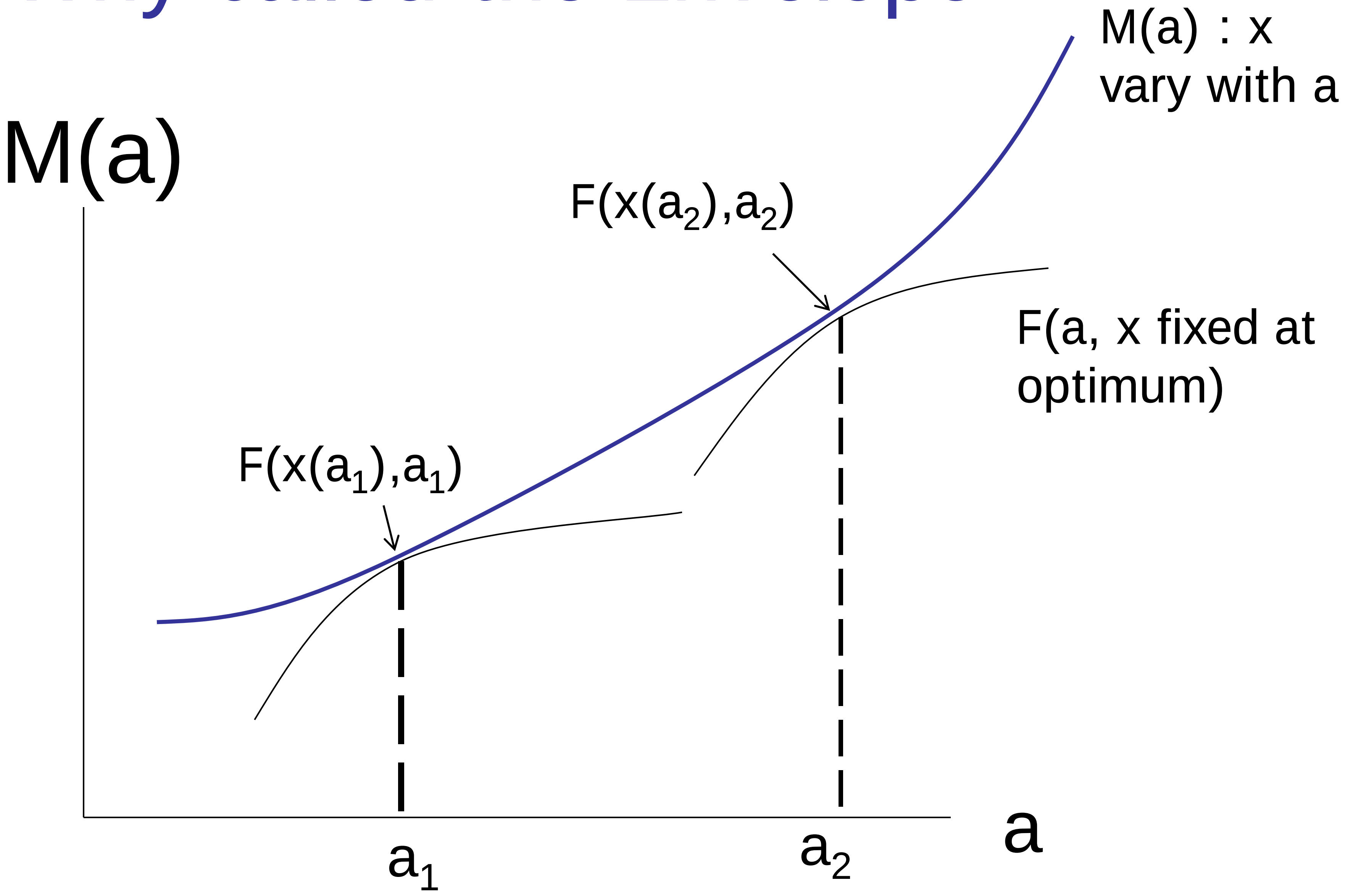
Value function

- From the previous example,
  $\partial U(a) / \partial a = a/8$.
- Envelop theorem,
  $\partial U(a) / \partial a = \partial L / \partial a$ evaluated at the solution of the original problem .
- From
  $\mathbf{L(x_1 , x_2 , \lambda) \equiv x_1 x_2 + \lambda [a - x_1 -4x_2]}$
- $\partial L / \partial a = \lambda^* = a/8$
- Thus, we save times to construct the value function and do the partial derivative.

Value function : Examples

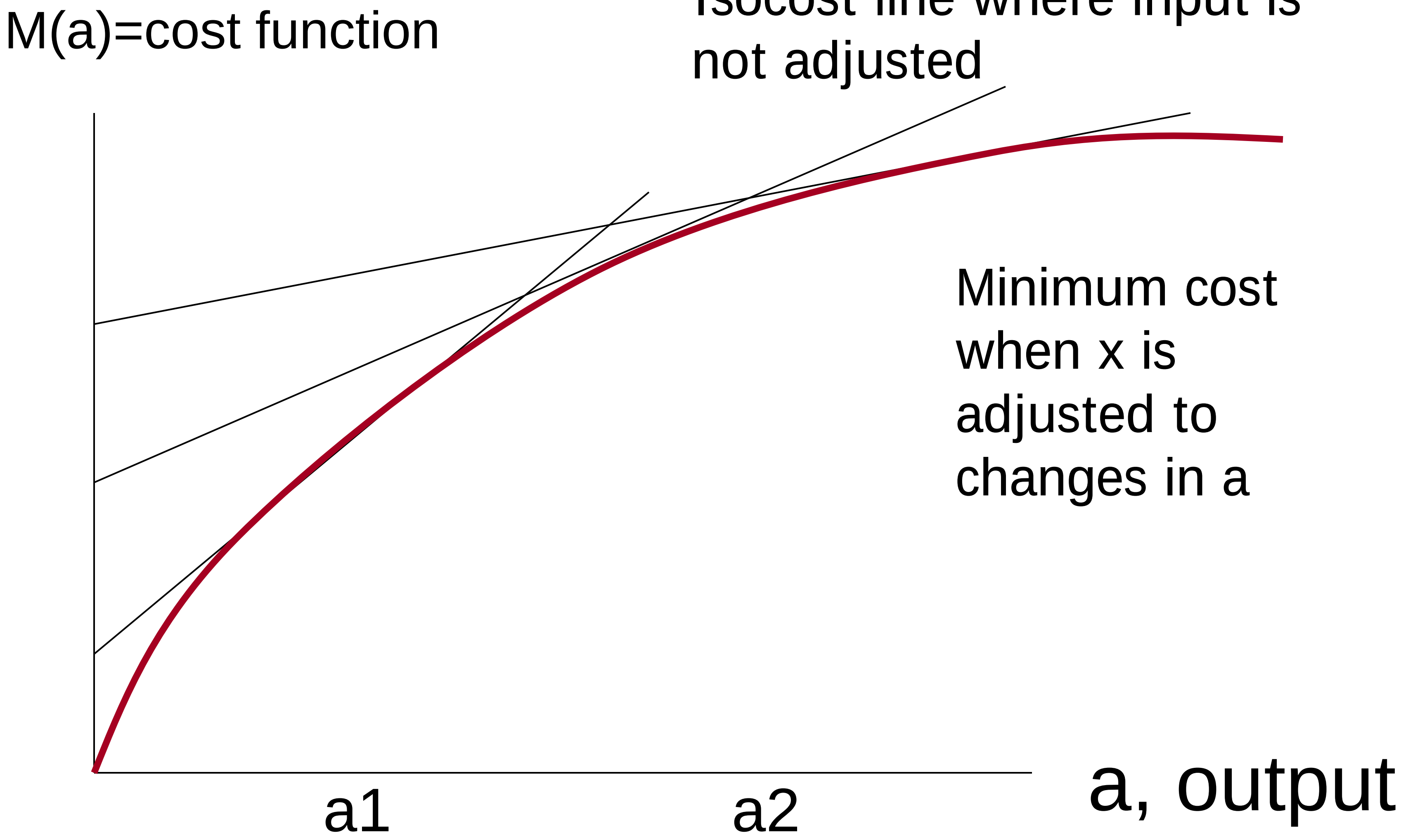
- In consumer problems: the indirect utility function, and expenditure function
- In producer problems: the profit function, and cost function.
- The value function is the envelope of the family of all original curves f with x fixed.

Why called the Envelope



- Try with max $f(x,a)=ax-x^2$.
- FOC: $a-2x=0$. So, $x^*=a/2$
- $M(a) = a(a/2)-(a/2)^2= a^2/4$.
- At $a= 1$, we have $x^*=0.5$, $M(a)=0.25$
- At $a=2$, we have new $x^*=1$, $M(a)=1$.
- What if at $a=2$, we keep use $x=0.5$, $f=2(0.5)-0.25=0.75$ (lower than 1)

Why called the Envelope



Isocost line where input is not adjusted

Minimum cost when x is adjusted to changes in a

Optimizations in Economics

- 1. Indirect Utility function:

  max $U(x_1, x_2)$ st. $p_1x_1 + p_2x_2 = I$.
- 2. Expenditure function:

  min $\mathbf{px}$ st. $u(\mathbf{x})=c$.
- 3. Profit function:

  max $p.y-\mathbf{wx}$ st. $f(\mathbf{x})=y$.
- 4. Cost function:

  min $w_1x_1 + w_2x_2$ st. $f(x_1, x_2)=c$

Applications of Envelope Theorem in Economics:

- 1. From indirect utility function, we can derive for Marshallian Demand fn.

$$\text{From } v(p, I)$$

$$\partial v / \partial I = \lambda, \quad \partial v / \partial p = -\lambda x, \quad \text{so}$$

$$x(p, I) = -\frac{\partial v / \partial p}{\partial v / \partial I} \quad (\because \frac{\lambda x}{\lambda})$$

Called "Roy's Identity"

Applications of Envelope Theorem in Economics:

- 2. From expenditure function, we can derive for Hicksian Demand fn.

$$\text{From}\; e(p,u) \equiv \min\; px \quad st \quad u(x) = c$$

$$\frac{\partial e}{\partial p} = x^{h}(p,u),$$

$$\text{Called "Shephard's Lemma"}$$

Applications of Envelope Theorem in Economics:

$$3.\ \text{From}\max\ py - wx\ \ \text{st}\ \ f(x) = y.$$

$$\text{FONC gives } x^*(p, w) \text{ or factor demand}$$

$$\text{Substitute } x^* \text{in production function,}$$

$$\text{we get an output supply function.}$$

$$\text{Substitute } x^* \text{ in the objective function,}$$

$$\text{we get } \pi(p, w) \text{ or "Profit function".}$$

Applications of Envelope Theorem in Economics:

- From profit function, we can derive for Input or factor demand and output supply.

$$\frac{\partial \pi}{\partial \mathrm{p}} = \quad \mathrm{y(p, w)}, \quad \text{output supply fn.}$$

$$\frac{\partial \pi}{\partial \mathrm{x}} = -\mathrm{x(p, w)}, \quad \text{factor demand fn.}$$

Applications of Envelope Theorem in Economics:

4. From cost function, we can derive for a Conditional Input demand.

$$\min \ wx \ \text{st} \ f(x) = y.$$

$$\text{FONC : conditional factor demand } x^{*}(w, y)$$

$$\text{Sub } x^{*}(w, y) \text{ in to the objective fn, we get } c(w, y)$$

$$\text{cost function}$$

$$\partial c / \partial x = x(w, y), \text{conditional factor demand}$$