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6.5 Exercises

1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02} f_2]$? Explain your answer.

- So, let S_i is the price of underlying asset at i, and let D_0 is the dividend at date 0

$$\to \quad S_0 = E_0[m_{01} D_1] + E_0[m_{02} S_2]$$
$$= D_0 + E_0[m_{02} S_2]$$

- And let F_{02} is a forward price, then we know that tha payoff is $S_2 - F_{02}$.

- By using stochastic discount factor approach, we found that

$$E_0[m_{02} f_2] = E_0[m_{02}(S_2 - F_{02})]$$
$$= E_0[m_{02} S_2] - E[M_{02} F_{02}]$$

noted that $S_0 = E_0[m_{01} D_1] + E_0[m_{02} f_2] = E_0[m_{02}(S_2 - F_{02})] = E[m_{02} S_2]$
and $E[m_{02} F_{02}] = E[m_{02}] F_{02} = R_f^{-2} F_{02}$

Therefore, $E_0[m_{02} f_2] = E[m_{02} S_2] - E[M_{02} F_{02}]$

$$= S_0 - D_0 - R_f^{-2} F_{02}$$

But we know that having no arbitrage means that $F_{02} = R_f^2 (S_0 - D_0)$ is satisfied, implying that

$$\boxed{E_0[m_{02} f_2] = 0}$$

2. Assume that there is an economy populated by infinitely-lived representative individuals who maximize the lifetime utility function

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right] \quad \text{— (1)}$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas (1978) endowment economy having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

→ Lucas model the price of risky asset is

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(C_t,t)}{U_c(C_0,0)} d_t\right]$$

→ From (1) it implies that $U(C_t,t) = -\delta^t e^{-ac_t}$, therefore $U_c(C_t,t) = a\delta^t e^{-ac_t}$. Moreover, since the economy is a Lucus endowment economy with one share per indivdual, so $C_t = d_1$.

As a result:

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(C_t,t)}{U_0(C_0,0)} d_t\right] = E_0\left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t-d_0)} d_t\right]$$

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3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

$$P_t = E_t\left[\sum_{t=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} d_{t+j}\right]$$

$$P_t/d_t = E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1}\left(\frac{d_{t+j}}{d_t}\right)\right]$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\ln(C_{t+j}/C_t)+\ln(d_{t+j}/d_t)}\right]$$

then

$$\ln(C_{t+j}/C_t) = j\cdot\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+j}$$

$$\ln(D_{t+j}/D_t) = j\cdot\mu_d + \sigma \sum_{i=1}^{j} \varepsilon_{t+j}$$

Therefore,

$$P_t/d_t = E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\left[j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+j}\right] + j\mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+j}}\right]$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{j[(\gamma-1)\mu_c+\mu_c] + \sum_{i=1}^{j}[(\gamma-1)\sigma_c \eta_{t+j} + \sigma_d \varepsilon_{t+j}]}\right]$$

$$= \sum_{j=1}^{\infty} \delta^j e^{j[(\gamma-1)\mu_c + \mu_d]} e^{\frac{j}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2 - 2(1-\gamma)\sigma_c\sigma_d\rho]}$$

$$= \sum_{j=1}^{\infty} e^{j[\ln\delta - (1-\gamma)\mu_c + \mu_d + \frac{1}{2}((1-\gamma)^2\sigma_c^2 + \sigma_d^2) - (1-\gamma)\sigma_c\sigma_c\rho]}$$

$$= \frac{1}{1-\delta e^{-(1-\gamma)\mu_c + \mu_d + \frac{1}{2}((1-\gamma)^2\sigma_c^2 + \sigma_d^2) - (1-\gamma)\sigma_c\sigma_d\rho}}$$

Hence :

$$P_t = d_t \frac{\delta e^{\alpha}}{1-\delta e^{\alpha}}, \text{ where } \alpha = \mu_d - (1-\gamma)\mu_c + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\rho\sigma_c\sigma_d$$

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4.a Show whether or not $p_t = f_t + b_t$ subject to the specifications in (1) and (2) is a valid solution for the price of the risky asset.

$$E_t[b_{t+1}] = \frac{R_f}{q_t} b_t q_t + E[e_{t+1}] q_t + (1-q_t) E_t[z_{t+1}] = R_f b_t$$

$\therefore$ It is a valid solution for the price of the risky asset.

4.b Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

When $E_t[b_{t+1}] = R_f b_t$, we know that

$$\lim_{i \to \infty} E_t[b_{t+i}] = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$$

For the limited liabilitie assets, it can't have a bubble path with a negative price, so we need to focus only bubbles with $b_t > 0$.

In this problem, the bubble component must be expected to increase infinitely. However, it can't be a rational expectation if the upper bound on the price of oil, as would be the case if there was a perfect substitute in perfectly elastic supply.

So, since P_t can't rise above P_{solar}, b_t can't rise above $P_{solar} - P_t^*$.

Therefore, a bubble path where b_t must increase to infinity can't occur.

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4.c Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

Ans: Rational speculative bubble cannot exist. Because at maturity, the bond's price must be $P_T = d_T$ and become 0 after date T therefore, the price cannot be expected to satisfy equation. Hence, a bubble path is invalid, and the only rational price is $P_t = P_t^*$.

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{C_s, \{\omega_{is}\}, \forall s,i} E_t \left[\sum_{s=t}^{T} \delta^s u(C_s) \right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

Ans: With the economy, asset prices cannot have the form $P_t = f_t + b_t$ because at date T, $P_T = f_T = d_T$ which is an assets's final dividend payment. Since $b_T = 0$ with certainty, the the bubble process $E_t[b_{t+1}] = \delta^{-1} b_t$ implies $E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$, or $b_{T-1} = 0$. A similar argument implies $b_t = 0$ for all previous dates, $t < T-1$.