EE422 (First semester 2013)
Mathematics For Economists 2

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Course materials and evaluation

- Main Texts: Sydsaeter et al. "Further Mathematics for Economic Analysis".
- Complementary texts: Simon and Blume; Chiang and Wainwright; Kamien and Schwartz; macro textbooks
- Grading: 4 HWs, midterm and final: 30:30:40

Objectives

- covers the mathematical structures commonly found in economics. It provides the basic mathematical background needed in quantitative economic analysis.
- The main focus of the course is the study of optimization theory (static and dynamic) and its applications to economics

Objectives

- Optimization concept.
- Static problem: Making choices with/without constraints
- Examples
  - Maximize utility under budget constraint
  - Minimize expenditure s.t. certain utility
  - Maximize profits under technology constraint
  - Minimize cost s.t. a given production level.
  - Maximize utility s.t. other utility and resource
- Tools used are Lagrangian method and Kuhn-Tucker
- Value function and Envelope Theorem

Objectives

- Statics: compare old and new equilibrium values of endogenous variables when one or more exogenous or parameters change.
- Dynamics: how the endogenous variables adjust to their new equilibrium values. Three aspects:
- Identification of steady-state or equilibrium values
- Analysis whether those steady-states are stable (otherwise our comparative static analysis will be meaningless)
- Description of type path of adjustment to new steady states, how quickly it is reached.

Objectives

- Two approaches:
- To model variables whose values change over time, we use difference equations or differential equations.
- Two general approaches for dynamic problem with constraints.
- Continuous time problem: Optimal control theory (Hamiltonian), and
- Discrete time problem: Dynamic programming (Bellman equation).

Example

- Example of a Dynamic problem : Price-adjustment model

$$Q_t^d = a - bP_t;\ \text{demand}$$

$$Q_t^s = \alpha + \beta P_t;\ \ \text{supply}$$

$$P_t - P_{t-1} = \lambda[Q_{t-1}^d - Q_{t-1}^s].$$

EXAMPLE 1 Logistic Growth

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

Objectives

- Dynamic problem with constraints: Economic growth model. The choice is the use of resources (or investment) in different time periods to maximize potential outputs.
- Maximizing a multi-period objective function of variables in many periods, subject to budget or resource constraints on these variables through time.
- Find optimal time path for every choice variable.

Example: Economic Growth

- Choices between current consumption and future consumption.
- By reducing consumption today, will lead to more resources or capital stock that can be used to produce consumption goods for future.
- What is the optimum consumption decision in each period? Or we try to find consumption function over time.

Example: Economic Growth

$$Ex1. \quad \max \int_0^T [u(c(t)e^{-\rho t}]dt \qquad \text{[Objective]}$$

$$st. \quad \dot{a}(t) = ra(t) - c(t), \qquad \text{[State eq.]}$$

$$a(0) = a_0, \qquad \text{[Initial cond.]}$$

$$a(T) = 0. \qquad \text{[Terminal cond.]}$$

$Control : c\,(consumption); \;\; State : a\,(assets). Time\,enters\,via$

$the\,discount\,factor(not\,appear\,in\,u, but\,outside).$

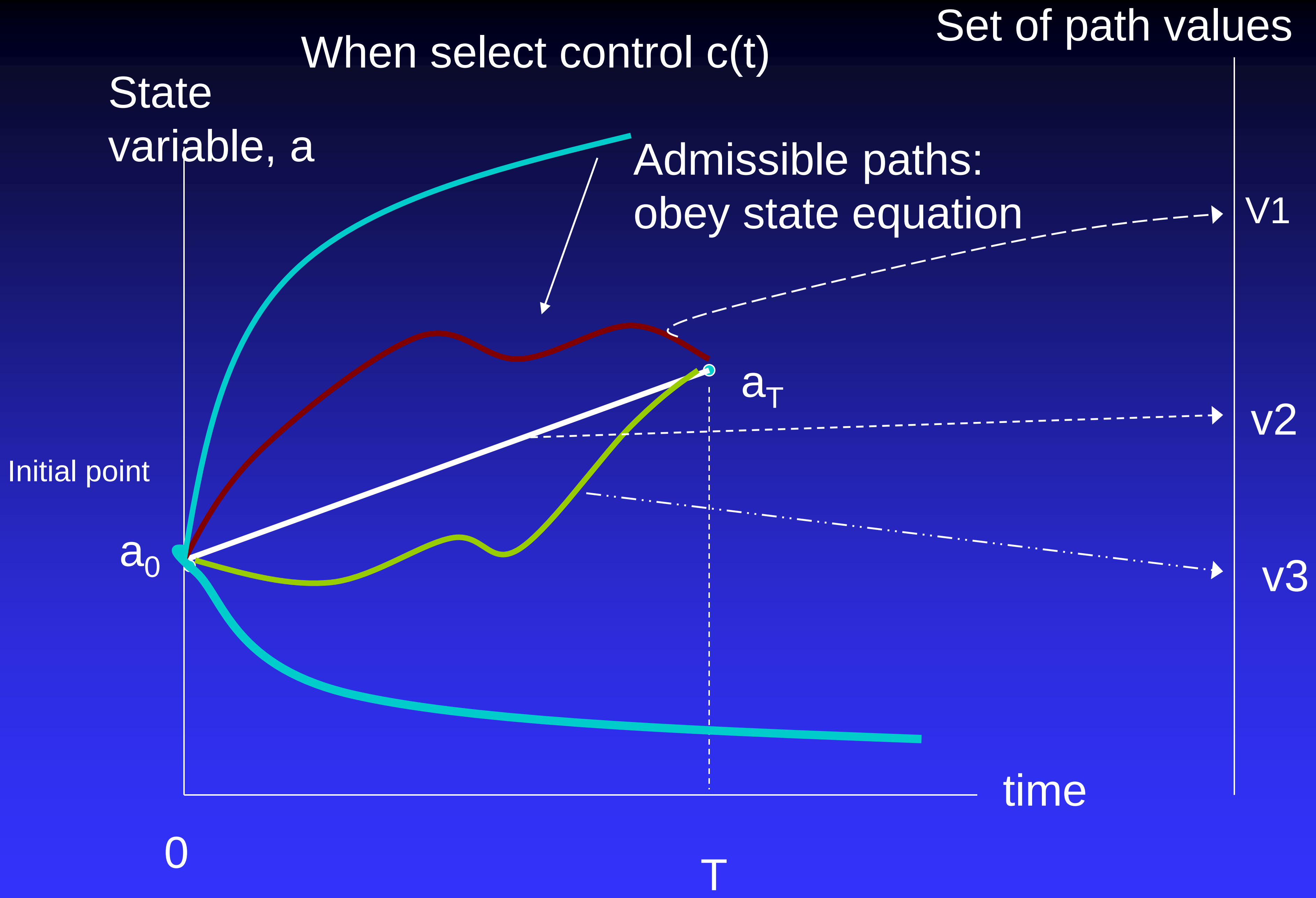
$e^{-\rho t}\; is\,a\,discount\,factor\,where\,\rho\,is\,a\,subjective\,discount\,rate.$

$$\cong \left(\frac{1}{1+\rho}\right)^t \; or\; \beta^t \; in\,discrete\,time.$$

$\dot{a}(t) \equiv da\,/\,dt, change\,in\,a\,at\,time\,t.[differential\,eq.]$

Common ingredients

- 1. Initial point and terminal point
- 2. Admissible paths from initial to terminal
- 3. Path values
- 4. Objective to maximize or minimize path value and get the optimal path.


Set of path values
When select control c(t)
State
variable, a
Admissible paths:
obey state equation
V1
a_T
v2
Initial point
a_0
v3
time
0
T

Example for DP.

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

$$s.t. \quad a_{t+1} = (1+r)a_t - c_t,$$

$$a_0 \text{ given.}$$

$$Bellman: V(a) = \max[\ln(c) + \beta V(a')]$$

$$st.\ a' = (1+r)a - c.$$

Topic 1 Preliminary Mathematics: some set theories and basic function in economics

Read chapter 2 in SYD or some good microeconomics textbooks

Mathematics is a language.

1.1 Convex Sets

- 1. We often assume this or convexity to guarantee that our analysis is tractable mathematically and the results are well-behaved or clear-cut.

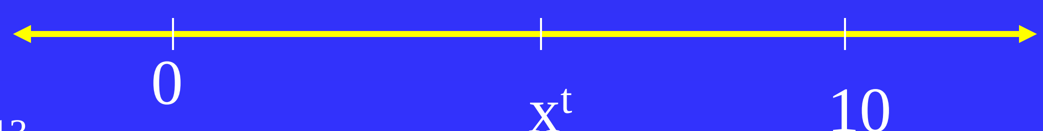
- 2. A set is convex if for any two points in the set, all convex combinations of those two points are also points in the same set.

Convex Sets

" S is a convex set if for all $\mathbf{x}^1$ and $\mathbf{x}^2$ belong to S, we have

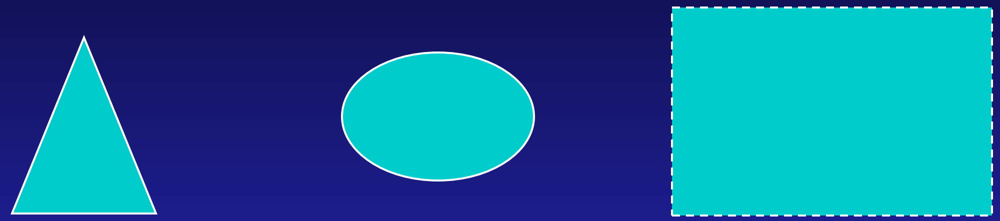
$t\,\mathbf{x}^1 + (1-t)\,\mathbf{x}^2 \in S$, for all $t \in 0 \le t \le 1$."

- Equivalently, a set is convex iff we can connect any two points in the set by a straight line that lie entirely within this set.
- 3. For example, consider two points in a real line, $\mathbb{R}$.

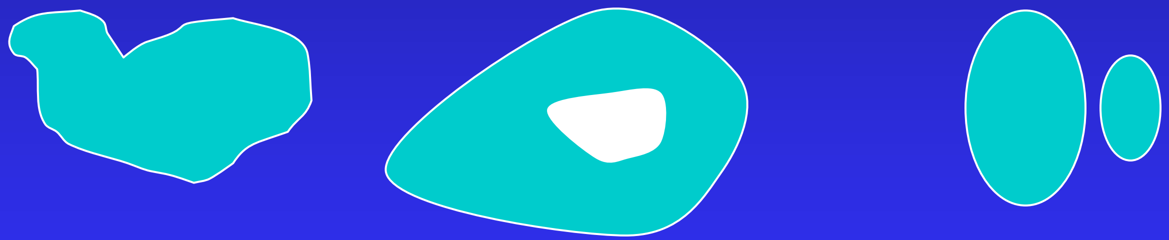


Convex Sets

- 5. Example of S in $\mathbb{R}^2$.



- It is not a convex set when it has a dense or holes, i.e. Heart, doughnut.



- Intuitively, a convex set must be connected without any holes and its boundary must not bend inwards.

Convex Sets

- 6. Intersections of convex sets is also convex.

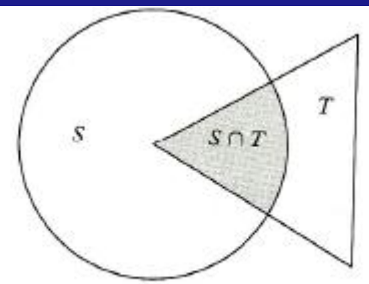


Figure 4 $S \cap T$ is convex, but $S \cup T$ is not.

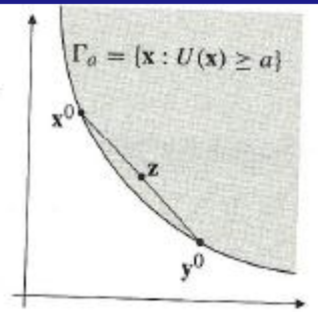


Figure 5 Γ_a is a convex set.

Open and Closed e Balls

- Definition
- 1. The open ball with center x^0 and radius $e > 0$ is the subset of points of x in R^n whose distance from x^0 is less than e. :

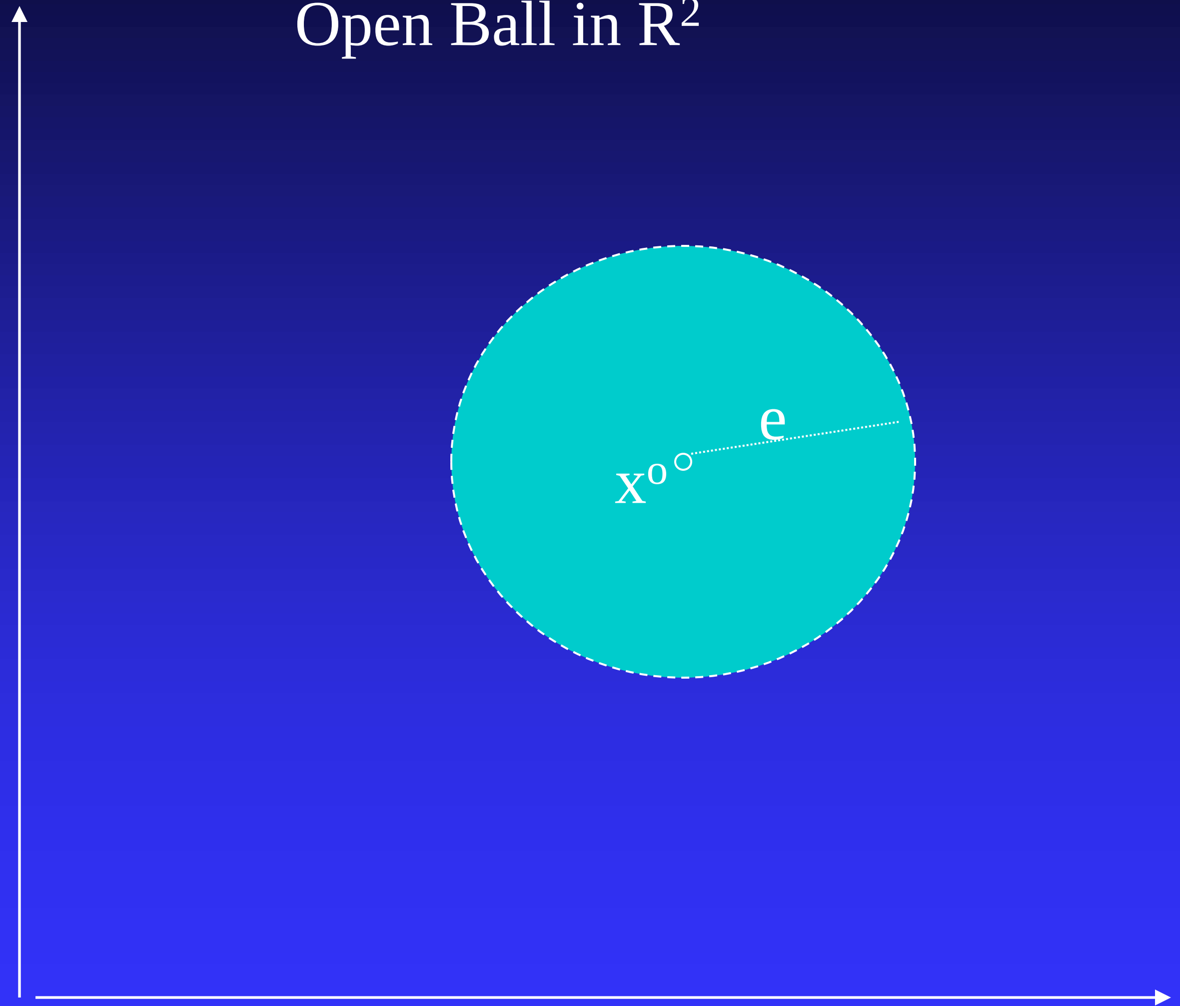
$$B_e(\mathbf{x}^o) = \{\mathbf{x} \in \mathbb{R}^n \;/\; \underbrace{d(\mathbf{x}^{\mathbf{0}}, \mathbf{x})}_{\substack{\text{distance} \\ \text{of two points}}} < e\}$$

- 2.The close ball, we add equality in the above definition.

Open and Closed e Balls

- In one dimension or on the real line, open ball with center x^o and radius e is just the open interval: $B_e(x^o) = (x^o - e, x^o + e)$
- And the correspondent close ball is the close interval $[x^o - e, x^o + e]$
- In 2 dimensions, open ball is an open disk in the plane.
- In 3 dimensions, open ball is a set of all points strictly inside the surface of a sphere. Think of radiation.

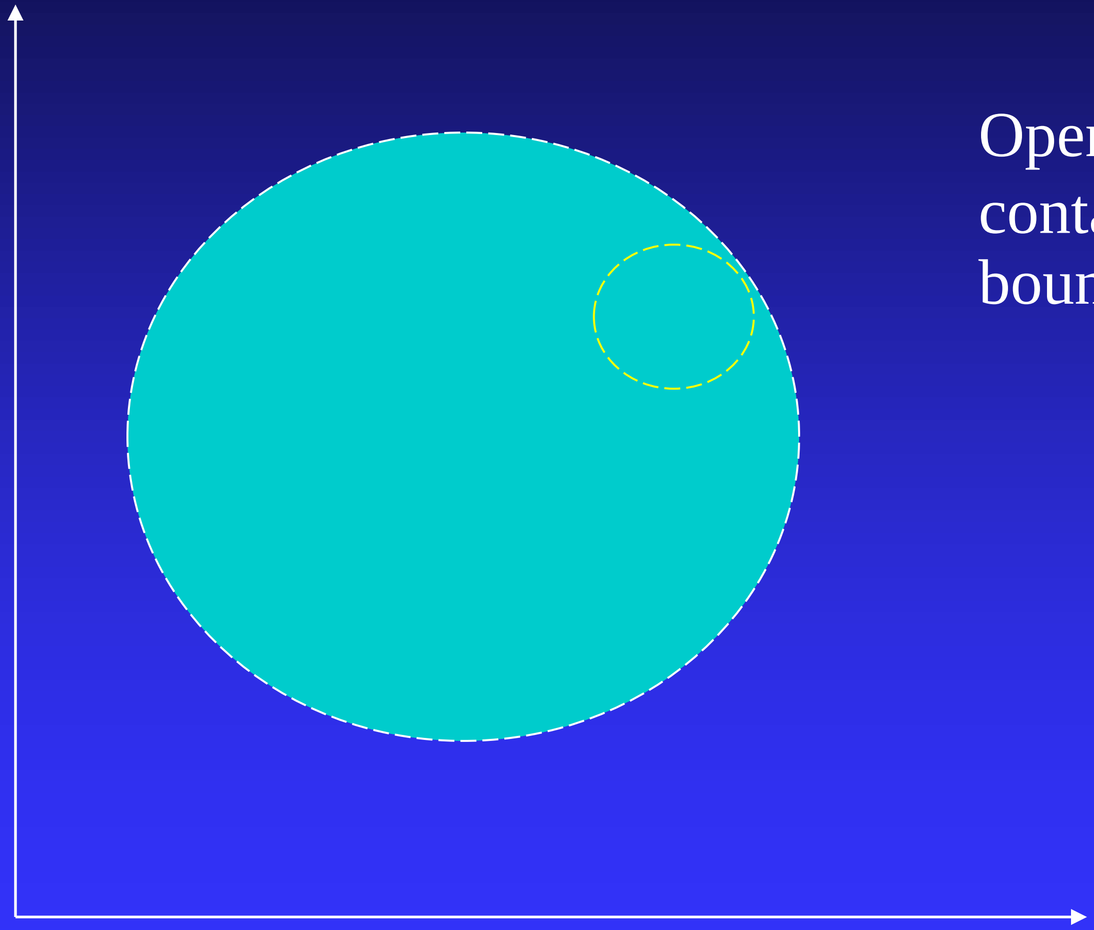
Open Ball in R^2



Open Sets

- Definition: S is an open set if for all x belongs to S, there exists some e > 0 such that $B_e(x) \subset S$.
- A set is open if for any points in the set, we can draw open ball (with any size) and all points in ball is still on the same set. In other words, all its members are interior points.
- Key idea. For all points, we can find open ball around it. Thus, for open interval (a,b), no matter how close to a or b, we can find some small open ball and that ball is contained entirely within the interval (a,b).
- Thus, every open interval on the real line is the open set. Open disks and open spheres are open sets. Also, the entire spaces R^2 , R^n and empty set are open sets.

An Open Ball is an open set



Open sets cannot contain their boundary points.

Open Sets

- 1. Every open interval on the real line, $(a, b) \subset \mathbb{R}$ is a open set.
- 2. In $\mathbb{R}^2$, an open ball is a disk containing set of points inside (excluding the circumference). (check that a line in $\mathbb{R}^2$ is not an open set, since a ball on any point on the line contains points not on the line)
- 3. In $\mathbb{R}^3$, an open ball is the set of points inside the sphere of radius ε. Again it is not closed.

Open Sets: properties

- 1. The whole space R^n and the empty set are both open.
- 2. Arbitrary unions of open sets are open.
- 3. The intersection of finitely many open sets is open.

Closed Sets

- 1. Consider the simplest case, a close interval, [a, b], in the real line is a closed set.
- The sets on both sides of this close interval are open interval, so they are open sets.

- The union of open sets is an open set. Thus, $[a, b]^c$ is an open set.
- 2. We call [a, b] a closed set if its complement is an open set. Or a set in R^n is closed iff its complement is open.

Closed Sets

- In higher dimensions, any closed ball is a closed set, a closed disk is a closed set, a closed sphere is a closed set.
- A set in R^n is closed if it contains all of the points on its boundary.
- X is called a boundary point of a set S in R^n if every ball around x contains points <u>in</u> S and points <u>not</u> in S. Think of house fence on the imaginary line.
- A set is open if it contains none of its boundary points.

Closed Sets: properties

- 1. The whole space R^n and the empty set are both closed.
- 2. Arbitrary intersections of closed sets are closed.
- 3. The union of finitely many closed sets is closed.
- 4. the union of infinitely many closed sets need not be closed. Ex. Infinite unions of A_i where $A_i = \{1 / i\}$

Some caution

- 1. the half open intervals are neither open nor close in R.
- 2.The whole space and the empty set are the only two sets in R^n that are both open and closed.
- 3.example of a set in R^2 that is neither open nor closed.

Bounded Sets

- Definition: A set S in R^n is bounded if it is entirely contained within some e ball.
- That is, S is bounded if there exists some e > 0 such that $S \subseteq B_e(x)$: we can always draw some ball entirely around it.
- Open interval and closed interval is bounded.
- Example of not bounded: (a, ∞), $(-\infty, \infty)$
- A sequence $\{x_k\}$ in R^n is bounded if the set $\{x_k / k = 1, 2, \ldots\}$ is bounded.
- Obviously, any convergent sequence is bounded.

Compact Sets

- 1. A set S in $\mathbb{R}^n$ is called *compact* iff it is *closed* and *bounded*.
- 2. Any open interval in R is not a compact set, since it is not closed.
- 3. An open ball in $\mathbb{R}^n$ is not compact, similarly.
- 4 $\mathbb{R}^n$ is not compact since it is not bounded.

Why a Compact or Convex Set

- 1. Some set is closed but not bounded. For example, $[2, \infty)$, set of positive integer, R^2, R^n.
- 2. We need a compact property for continuity property and we need this this continuity to prove the existence of the maximum, known as the Weierstrass Theorem.
- 3. As for the convex set, we use this in General equilibrium theory and the Hyperplane theorem.

Real-valued functions

- f is a real-value function if it maps elements of its domain into the real line.
- Suppose D in $\mathbb{R}^n$, $f : D \rightarrow \mathbb{R}$
- Ex. $F(x_1, x_2) = ax_1 + bx_2$
- Ex. $U(x_1, x_2) = x_1x_2$

1.2 Level sets

- This allows us to study functions of three variables by looking upon sets in the two dimensional plane.
- 1. A level set is the set of all elements in the domain of a function that map into the same number in the range.
- 2. L(y=10) is a level set of the real-valued function $f : D \to \mathbb{R}$ iff $L(\ y=10) = \{\mathbf{x} \mid \mathbf{x} \in D,\ f(\mathbf{x}) = 10\ \}$.

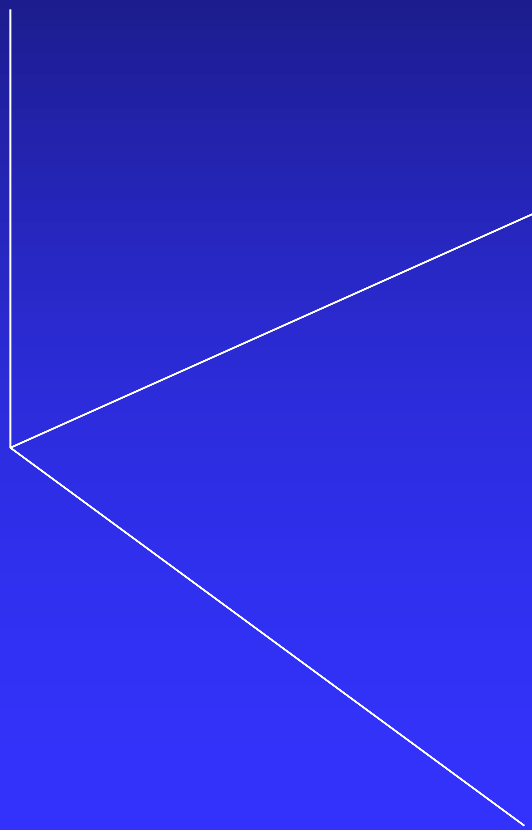
Level sets

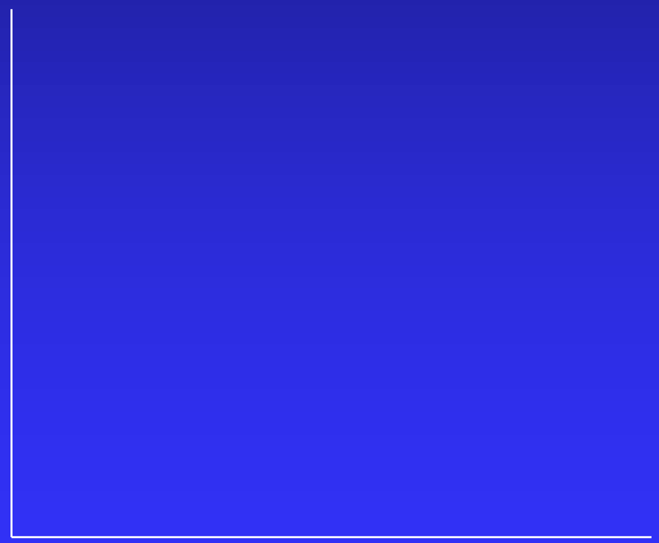
Ex. An isoquant is a set that all input bundles producing exactly 10 units of output.

- 3. Note that since f is a function-- that is it assigns a single number in the range—therefore, two difference level sets will never cross each other.

Level Sets

- Draw a picture of the function and its level sets





Level sets

- 4. Draw level sets in $\mathbb{R}^2$ and define the superior set,

  $S(y=10) \equiv \{\mathbf{x} | \mathbf{x} \in D, f(\mathbf{x}) \geq 10\}$, for level y=10.

Ex. An input requirement set is a set of all input bundles that produce at least y units of output.

- 5. When f(**x**) is increasing, then S(y=10) will always lie on and above the level set L(y=10)

Level sets

- 6. Note that (a) f is increasing if $f(\mathbf{x}^0) \geq f(\mathbf{x}^1)$, for all $\mathbf{x}^0 \geq \mathbf{x}^1$. (notice vector notation here).
- 7. Think of the level sets when f is decreasing. For example, $v(p_1, p_2, y) = y / p_1^a p_2^{1-a}$. Find the lower contour sets.