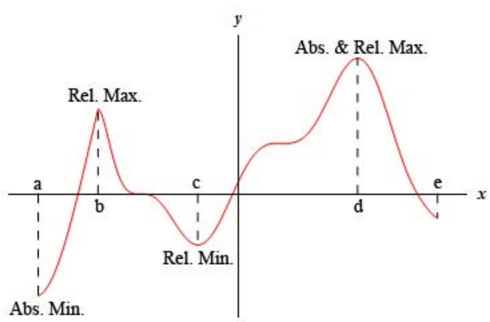
Differentiation: Application

- Extrema of Functions
- Graphing using the First Derivative and the Second Derivative
- Optimization

1 Extrema of Functions

Definition 1.1 (Extrema)**.** Let $f(x)$ be a function defined on a domain $\mathcal{D}$.

- $f(x)$ has an **absolute (or global) maximum** at $c \in \mathcal{D}$if $f(c) \geq f(x)$ for every $x \in \mathcal{D}$.
- $f(x)$ has a **relative (or local) maximum** at $c \in \mathcal{D}$ if $f(c) \geq f(x)$ for every $x \in \mathcal{D}$ in some open interval around c.
- $f(x)$) has an **absolute (or global) minimum** at $c \in \mathcal{D}$at if $f(c) \leq f(x)$ for every $x \in \mathcal{D}$.
- $f(x)$ has a **relative (or local) minimum** at $c \in \mathcal{D}$ if $f(c) \leq f(x)$ for every $x \in \mathcal{D}$ in some open interval around c.



Theorem 1.1 (**Extreme Value Theorem**)**.** A function f **continuous** on a **closed** interval $[a, b]$ always has an absolute maximum and an absolute minimum on the interval.

Definition 1.2. When an absolute extremum of a function occurs at an endpoint of a closed interval I, we say that it is an **endpoint extremum**.

Example 1.1. A Continuous Function Defined on a Closed Interval

(a) $f(x) = x^2$ defined on the closed interval $[1, 2]$ has the absolute maximum $f(2) = 4$ the absolute minimum $f(1) = 1$. Each of these points $(2, 4)$ and $(1, 1)$ is called an **endpoint extremum**.

(b) $f(x) = x^2$ defined on the closed interval $[-1, 2]$ has the absolute maximum $f(2) = 4$ the absolute minimum $f(0) = 0$.

Remarks

1. The theorem above tells us that if $f(x)$ is **continuous** on $\mathcal{I}$ that is **a closed interval**, then $f(x)$ is **guaranteed** to have an absolute extremum (see Example 1.1).

2. For a function that is **not continuous** on an interval $\mathcal{I}$, it may or may not have an absolute extremum (even though $\mathcal{I}$ is a closed interval).

   E.g. The following function $f(x)$ defined below on $[-2, 2]$ has no extrema.

   $$f(x) = \begin{cases} \frac{1}{x} & , \quad -2 \le x < 0 \\ 0 & , \quad x = 0 \\ \frac{1}{x} & , \quad 0 < x \le 2 \end{cases}$$

3. For a function $f(x)$ defined on an open interval $\mathcal{I}$, e.g. $(a, b], (-\infty, b]$ or $[a, \infty)$, $f(x)$ **may or may not** have an extrema on that open interval $\mathcal{I}$ (even though f is continuous).

   E.g. for $f(x) = x^2$ defined on the open interval $(1, 2)$, f has no absolute extremum. Note that $f(1)$ and $f(2)$ are not defined in this case.

The next example shows that for a continuous function, it is possible to have absolute extrema on an **open interval**.

Example 1.2. (Exercise) Find all the absolute extrema (absolute maxima and minima) of the function $f(x) = \sin(x)$ on the set of all real numbers $(-\infty, \infty)$.
[Ans: set of absolute maxima $\{x = \pi/2 + 2n\pi | n \in \mathbb{Z}\}$; set of absolute minima $\{x = 3\pi/2 + 2n\pi | n \in \mathbb{Z}\}$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ is the set of integers.]

Definition 1.3. (Critical Numbers) Let $f(x)$ be a function defined on a domain $\mathcal{D}$. A **critical number** (or **critical point**) of f is a number $c \in \mathcal{D}$ such that

- $f'(c) = 0$, or
- $f'(c)$ does not exist.

Example 1.3. Find the critical numbers of $f(x) = (x-1)^{2/3}$.
Solution:[Ans: $x = 1$]
-First, note that the domain of f is: $\mathcal{D} =$
-Find $f'(x)$:
$f'(x) =$

- For what value of x that $f'(x) = 0$?

- For what value of x that $f'(x)$ does not exist ?

Hence the critical number is .... ■

Example 1.4. Find the critical numbers of $f(x) = \frac{x^2}{x-1}$.

Solution:[Ans: $x = 0, 2$ note that $x = 1$ is not a critical number]

-First, note that the domain of f is: $\mathcal{D} =$
-Find $f'(x)$:
$f'(x) =$

- Notice that $f'(x) = 0$ when ....

- For what value of x that $f'(x) = 0$ does not exist ?
Is that value of x in $\mathcal{D}$?

Notice for **any** $x \in \mathcal{D}$**, that** $f'(x)$ **is well-defined**.

Hence the critical numbers are........... ■

Example 1.5. (Exercise) Find the critical numbers of

$$f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 4.$$

Solution:[Ans: $x = 1, 2$]

Example 1.6. (Exercise) Find all the critical numbers of $f(x) = x\ln(x)$.
Solution:[Ans: $x = 1/e$]

1.1 Finding Absolute Extrema from Critical Numbers

From Extreme Value Theorem (Theorem 1.1), a given continuous function f defined on a closed interval $[a, b]$ always has the absolute maximum and minimum on $[a, b]$. The following theorem tells us where these extrema can occur on $[a, b]$.

Theorem 1.2. If $f(x)$ is continuous on $[a, b]$, the **an absolute extremum** occurs either at

- Endpoints: $x = a$ or $x = b$, or
- Critical number $x = c$: $f'(c) = 0$, or $f'(c)$ does not exist.

Steps for finding extrema of a continuous function $f(x)$ on a closed interval $[a, b]$:

(1) Find all critical numbers $x = c_1, c_2, \ldots, c_n$:

$f'(c_j) = 0$, or $f'(c_j)$ does not exist, for $j = 1, \ldots, n$

(2) Compute

(i) $f(a)$ and $f(b)$ (Endpoints)
(ii) $f(c_1), f(c_2), \ldots, f(c_n)$ (Critical numbers).

and compare these values to find the absolute maximum and minimum.

Example 1.7. Find the absolute extrema of $f(x) = x^3 - 3x^2 - 24x + 2$ on the following intervals.

(a) $[-3, 1]$ (b)$[-3, 8]$

Solution: [Note that $f(-3) = 20$,$f(-2) = 30$,$f(1) = -24$, $f(4) = -78$, $f(8) = 130$]

1.2 Increasing/Decreasing

Definition 1.4 (Increasing/Decreasing Function)**.** Let $f : X \to Y$ be a function $y = f(x)$, $X, Y \subseteq \mathbb{R}$. Let $S \subseteq X$.

- f is increasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is decreasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Theorem 1.3 (Test for Increasing/Decreasing)**.** Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b).

(i) If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.

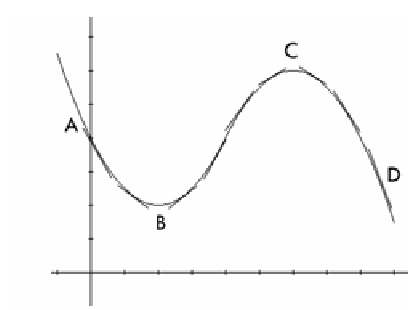
(ii) If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

(iii) If $f'(x) = 0$ for all x in $[a, b]$, then f is a constant on the interval.

Proof. Recall that (iii) is the same as Theorem **??**. Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. Since $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then $f(x)$ also is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), From the Mean Value Theorem, there is a number $c \in (x_1, x_2)$ such that $\boxed{f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}.}$ Note that $x_2 - x_1 > 0$ since $x_1 < x_2$.
(i) By hypothesis, $f'(c) > 0$ and therefore, $f(x_2) - f(x_1) > 0 \Longrightarrow f(x_1) < f(x_2)$ for $x_1 < x_2$. Since x_1 and x_2 can be arbitrarily chosen from $[a, b]$, $f(x)$ is increasing on $[a, b]$ by the definition.
(ii) By hypothesis, $f'(c) < 0$ and therefore, $f(x_2) - f(x_1) < 0 \Longrightarrow f(x_1) > f(x_2)$ for $x_1 < x_2$. Since x_1 and x_2 can be arbitrarily chosen from $[a, b]$, $f(x)$ is decreasing on $[a, b]$ by the definition.

□

The intervals of increase and decrease will occur between points where $f'(x) = 0$ or $f'(x)$ is undefined (i.e. critical points). However, these points are not necessarily critical numbers because we include x even if it is not in the domain of f. We simply want to find the intervals of increase and decrease around x, even if the function is not defined at that point.



The graph illustrates this theorem. From A to B, the slope of the tangent lines are all negative, so the derivative, $f'(x)$ is negative from A to B. The theorem above states that the function is decreasing from A to B. The graph shows that the values of the function are decreasing between A and B. Similarly, the function is also decreasing between C and D. From B to C however, the slopes of the tangent lines are positive. Therefore, the derivative is positive from B to C. The graph shows that the values of the function are increasing between B and C.

Example 1.8. Find the interval on which the function f is increasing and the interval on which f is decreasing.
(a) $f(x) = x^3 - 3x^2 - 24x$ [Ans: increasing on $(-\infty, -2) \cup (4, \infty)$ and decreasing on $(-2, 4)$]
(b) $f(x) = \frac{x^2-3}{x^2+1}$ [Ans: $f'(x) = \frac{8x}{(x^2+1)^2}$; increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$]

1.3 Finding Relative Extrema from Critical Numbers

Relative extrema: (defined again below) of f can be identified by using the first derivatives and the second derivatives from the following theorems.

Definition 1.5. [Relative Extrema: Maximum/Minimum]
- A number $f(c)$ is a **relative maximum** of a function f if $f(x) \leq f(c)$ for every x in *some* open interval that contains c.

- A number $f(c)$ is a **relative minimum** of a function f if $f(x) \geq f(c)$ for every x in *some* open interval that contains c.

Theorem 1.4. (Relative Extrema Occur at Critical numbers) [Fermat's Theorem] If a function f has a relative extremum at $x = c$, then c is a critical number.

A critical point of f may not always be a relative extremum. For example, $f(x) = x^3$ or $f(x) = x^{1/3}$. When $f(x) = x^3$, $f'(x) = 3x^2$. We see that $f'(x) = 0$ when $x = 0$. But $x = 0$ is not a relative (local) extremum.
From the previous theorem, we can find an extremum by first find all critical numbers c of f first. Then, we can use two following approaches to identify if each critical number gives maximum, minimum, or it does not give any extremum.

- The First Derivative Test for Relative Extrema
- The Second Derivative Test for Relative Extrema

1.3.1 The First Derivative Test for Relative Extrema

Theorem 1.5. (First Derivative Test for Relative Extrema)

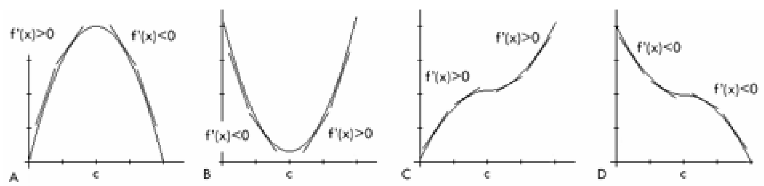
Let f be continuous on $[a, b]$ and differentiable on (a, b) except possibly at the critical number c.

(i) If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative **maximum**.

(ii) If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative **minimum**.

(iii) If $f'(x)$ has the same algebraic sign on each side of c, then $f(c)$ is **not an extremum**.

The graphs below illustrate the first derivative test.



When there is only one critical number for the function f on its entire domain, there is an important relationship between the relative extremum and absolute extremum as considered in the following theorem.

Theorem 1.6. (The Sole Critical Number Test) Suppose c is the only critical number of a function f with in an interval I. If it is proved that $f(c)$ is a *relative* extremum, then $f(c)$ is an *absolute* extremum.

Example 1.9 (Exercise)**.** Use the First derivative test to determine all the relative extrema of the following functions.
(a) $f(x) = x^2 + x - \ln(x)$
(b) $f(x) = -x^{5/3} + 5x^{2/3}$
(c) $f(x) = \sqrt{x}e^{-x/2}$ [Ans: increasing on $(0, 1)$ and decreasing on $(1, \infty)$]

Example 1.10. Let $f(x) = x^3 - 3x^2 - 9x + 2$. Find the interval on which the function f is increasing and the interval on which f is decreasing. Use the First derivative test to specify all the relative extrema. [Ans: Max at $x = -1$ and Min at $x = 3$]

1.3.2 Concavity & The Second Derivative Test for Relative Extrema

Concavity and the point of inflection: (defined below) of f can be identified by using the the second derivatives from the following theorem.

Definition 1.6 (Concavity: concave up/concave down)**.** Let f be a differentiable function on an interval (a, b).

(i) If f' is an increasing function on (a, b), then the graph of f is **concave up** on the interval.

(ii) If f' is an decreasing function on (a, b), then the graph of f is **concave down** on the interval.

Definition 1.7 (Point of Inflection)**.** Let f be a continuous on an interval (a, b) containing the number c. A point $(c, f(c))$ is a **point of inflection** of the graph f if there is a tangent line at $(c, f(c))$ and the graph changes concavity at this point.

Theorem 1.7. (Test for Concavity) Let f be function for which f'' exists on (a, b).

If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave up on (a, b).

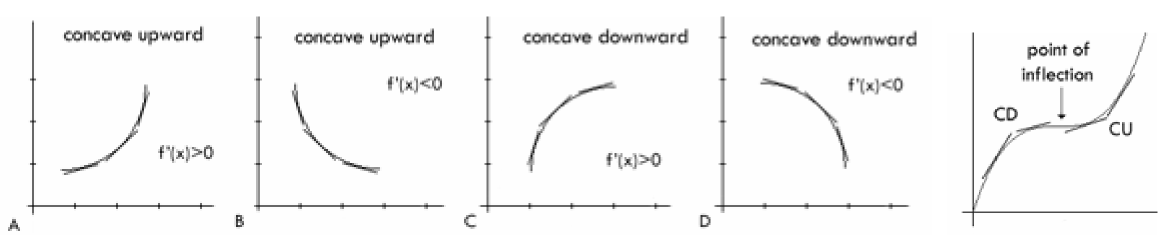
(ii) If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave down on (a, b).

Theorem 1.8. (Point of Inflection) If $(c, f(c))$ is a point of inflection for the graph of a function f, then either

$$f''(c) = 0 \qquad \text{or} \qquad f''(c) \text{ does not exist.}$$

Points of inflection may occur at points where $f''(x) = 0$ or $f''(x)$ is undefined, where x is in the domain of f. We must test the concavity around these points to determine whether they are points of inflection.

The graph below illustrate concavity of different curves and show a curve with a point of inflection.



Theorem 1.9. (Second Derivative Test for Relative Extrema)

Let f be a function for which the second derivative f'' exists on an interval (a, b) that contains the critical number c.

(i) If $f''(c) > 0$, then $f(c)$ is a relative **minimum**.

(ii) If $f''(c) < 0$, then $f(c)$ is a relative **maximum**.

(iii) If $f''(c) = 0$, the test fails and $f(c)$ may or may not be a relative extremum. In this case, use the First derivative Test.

Example 1.11. Let $f(x) = x^3 - 3x^2 - 9x + 2$. Determine the concavity and the point of inflection (if any) for f. Use the Second derivative test to specify all the relative extrema.
[Ans: Max at $x = -1$ and Min at $x = 3$]

Example 1.12. (Exercise) Let $f(x) = x^4 - 4x^3 + 10$.
(a) Find the interval on which the function f is increasing and the interval on which f is decreasing. Use the First derivative test to specify all the relative extrema.
(b) Determine the concavity and the point of inflection (if any) for each function. Use the Second derivative test to specify all the relative extrema. [Ans: No Max, Min at $x = 3$]

2 Graphing Functions

Let f be a relation from X to Y where $X, Y \subseteq \mathbb{R}$. The formal definition of the curve of f is given below.

Definition 2.1 (Graph/Curve)**.** Let X and Y be subsets of $\mathbb{R}$. For a relation $f : X \to Y$, each ordered pair $(x, f(x))$, $x \in X$, can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph or curve of** f.

We will consider a relation f that is a *continuous* and *differentiable* **function** defined from X to Y where $X, Y \subset \mathbb{R}$. To sketch a curve for f, it is helpful to identify the following characteristics of the curve.

(1) Domain and Range (undefined points)

(2) Intercepts: the x-and y-intercepts

(3) Asymptotes: Vertical/Horizontal/Slant Asymptotes

(4) Increasing/Decreasing Intervals

(5) Local (Relative) Maximum/Minimum

(6) Concavity and Points of Inflection

2.1 Review

(1) **Domain & Range** of f: Find the set of all values x such that f is well-defined and the corresponding y. This will be useful when finding vertical asymptotes and determining critical numbers.

(2) **Intercepts:** Find the x-and y-intercepts for $(x, y) \in f$, if possible.

- To find the x-intercept, set $y = 0$ and solve the equation for x.
- To find the y-intercept, set $x = 0$ and solve the equation for y.

(3) Asymptotes of a curve. We will consider two types of asymptotes.

- **Vertical asymptote**: By using the equation in the form $y = f(x)$, $\boxed{x = a}$ is a vertical asymptote if

$$\boxed{\begin{array}{l}\lim_{x\to a} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \infty \quad \text{or} \\ \lim_{x\to a} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty.\end{array}}$$

- **Horizontal asymptote**: By using the equation in the form $y = f(x)$, $\boxed{y = b}$ is a horizontal asymptote if $\boxed{\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.}$

Example 2.1 (Exercise)**.** Consider a function $f(x) = \dfrac{x^2 + 5x + 4}{x^2}$.

1. Find domain and range of f.
2. Find x-intercepts and y-intercepts.
3. Find the asymptotes for f (if any).

Summary

- First, draw dashed lines for the asymptotes of the function.
- Then plot the x-and y-intercepts, maximum and minimum points and points of inflection on the graph.
- Sketch the curve between the points, using the intervals of increase and decrease and intervals of concavity.
- Be sure that the graph behaves correctly when approaching asymptotes.

Example 2.2. Consider the functions

(a) $f(x) = 4x^4 - 4x^2$, (b) $f(x) = 3x^{2/3} - x$, (c) $f(x) = \dfrac{x^2 - 3}{x^2 - 1}$

(1) Find the increasing and decreasing intervals of f
(2) Find the local maximum and minimum values of f.
(3) Find the concavity and the inflection points of f.
(4) Sketch the graph of f by using the above information.

Solution: (a) $f(x) = 4x^4 - 4x^2$.

[Continued]

[Continued] (b) $f(x) = 3x^{2/3} - x$

[Continued] (c) $f(x) = \dfrac{x^2 - 3}{x^2 - 1}$,

Note: $f'(x) = \dfrac{4x}{(x^2 - 1)^2}$, $\quad f''(x) = \dfrac{-4(3x^2 + 1)}{(x^2 - 1)^3}$

[Continued]

Example 2.3. (Exercise) Graph the following functions.
(a) $f(x) = \frac{x^2-3}{x^2+1}$
(b) $f(x) = x^2 + x - \ln(|x|)$
(c) $f(x) = -x^{5/3} + 5x^{2/3}$
(d) $f(x) = x^4 - 4x^3 + 10$
(e) $f(x) = 2\cos(x) - \cos(2x)$

3 Optimization: Maximum/Minimum Problems

GUIDELINES FOR SOLVING MAX./MIN. PROBLEMS

1. Read each problem slowly and carefully. Read the problem at least three times before trying to solve it. Sometimes words can be ambiguous. It is imperative to know exactly what the problem is asking. If you misread the problem or hurry through it, you have NO chance of solving it correctly.

2. If appropriate, draw a sketch or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out your thoughts.

3. Define variables to be used and carefully label your picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.

4. Write down all equations which are related to your problem or diagram. Clearly denote that equation which you are asked to maximize or minimize. Experience will show you that MOST optimization problems will begin with two equations. One equation is a ”constraint” equation and the other is the ”optimization” equation. The ”constraint” equation is used to solve for one of the variables. This is then substituted into the ”optimization” equation before differentiation occurs. Some problems may have NO constraint equation. Some problems may have two or more constraint equations.

5. Before differentiating, make sure that the optimization equation is a function of only one variable. Then differentiate using the well-known rules of differentiation.

6. Verify that your result is a maximum or minimum value using the first or second derivative test for extrema.

Example 3.1 (Exercise)**.**

1. A box with a square base is to be constructed. Suppose that only 10 m^2 of material is available to use in construction of the box. Assume that all the material is used in the construction process. Determine the maximum volume that the box can have.

2. We want to print a poster that has a total area of 200 in^2 with 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. Determine the dimensions of the poster that will give the largest printed area.

3. Determine the area of the largest rectangle that can be inscribed in a circle of radius 4 cm.

4. Find the point on the graph of $y^2 = 2x$ closest to $(2, 0)$.

Example 3.2. We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Example 3.3. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50ft^3, determine the dimensions that will minimize the cost to build the box. What is the corresponding minimum cost of this box?

Example 3.4.

At midnight, ship A is 50 km north of ship B. Ship A is sailing south at 20 km/h. Ship B is sailing west at 10 km/h. At what time will the distance between the ships be a minimum?