Answer Keys Chp5

1. a) $Z = -2x^2 + y^2$ subject to $2x - y = 1$

   $L(x, y, \lambda) = -2x^2 + y^2 + \lambda(1 - 2x + y)$

   1. <u>FOC:</u> $L_x = -4x + \lambda(-2) = 0$ -----1)

      $L_y = 2y + \lambda(1) = 0$ -----2)

      $L_\lambda = 1 - 2x + y = 0$ -----3)

      Solve 1),2) and 3) $x=1$, $y=1$ and $\lambda = -2$

   2. <u>SOC:</u> $[H] = \begin{bmatrix} 0 & 2 & -1 \\ 2 & -4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

      $|H_2| = \begin{vmatrix} 0 & 2 & -1 \\ 2 & -4 & 0 \\ -1 & 0 & 2 \end{vmatrix} = -4 < 0$ (Minimum)

   So, at $x=1$, $y=1$ gives Z minimum $= -2(1)^2 + (1)^2 = -1$.

   b) $Z = xy$ subject to $15x + 5y - 150 = 0$

   $L(x, y, \lambda) = xy + \lambda(150 - 5x - 5y)$

   1. <u>FOC:</u> $L_x = y + \lambda(-5) = 0$ -----1)

      $L_y = x + \lambda(-5) = 0$ -----2)

      $L_\lambda = 150 - 5x - 5y = 0$ -----3)

      Solve 1),2) and 3) $x=15$, $y=15$ and $\lambda = 3$

   2. <u>SOC:</u> $[H] = \begin{bmatrix} 0 & 5 & 5 \\ 5 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix}$

      $|H_2| = \begin{vmatrix} 0 & 5 & 5 \\ 5 & 0 & 1 \\ 5 & 1 & 0 \end{vmatrix} = 50 > 0$ (Maximum)

   So, at $x=15$, $y=15$ gives Z maximum = 225

   c) $Z = 7 - y + x^2$ subject to $x + y = 0$

   $L(x, y, \lambda) = 7 - y + x^2 + \lambda(-x - y)$

   1. <u>FOC:</u> $L_x = 2x + \lambda(-1) = 0$ -----1)

      $L_y = -1 + \lambda(-1) = 0$ -----2)

      $L_\lambda = -x - y = 0$ -----3)

Solve 1),2) and 3) x=$\frac{-1}{2}$, y=$\frac{1}{2}$ and λ= - 1

2.<u>SOC:</u> $[\mathrm{H}]= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix}$ = 2 >0 (Maximum)

So, at x=$\frac{-1}{2}$, y=$\frac{1}{2}$ gives Z maximum = $7 - \frac{1}{2} + (-\frac{1}{2})^2 = \frac{27}{4}$

d) Z=x+2y+3w+xy-yw subject to x+y+2w=10

L (x,y,λ) = x+2y+3w+xy-yw + λ(10 – x – y – 2w)

1.<u>FOC:</u> Lx= 1+y+ λ(-1)=0 -----1)

Ly= 2+x – w + λ(-1)=0 -----2)

Lw= 3– y + λ(-2)=0 -----3)

Lλ= 10 – x – y – 2w =0 -----4)

Solve 1),2) and 3) x=$\frac{25}{9}$, y=$\frac{1}{3}$, w=$\frac{31}{9}$and λ= $\frac{4}{3}$

2.<u>SOC:</u> $[\mathrm{H}]= \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 2 & 0 & -1 & 0 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ = 2 >0 (Maximum)

$|H_3| = \left\|\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 2 & 0 & -1 & 0 \end{bmatrix}\right\|$ = 9 >0 (Maximum)

So, at x=$\frac{25}{9}$, y=$\frac{1}{3}$, w=$\frac{31}{9}$and λ= $\frac{4}{3}$ is inflection point.

e) Z= xy subject to $x^2+y^2=16$ where x,y >0

L (x,y,λ) = xy+ λ(16 - x^2 - y^2)

1.<u>FOC:</u> Lx= y + λ(-2x)=0 -----1)

Ly= x + λ(-2y)=0 -----2)

$L\lambda = 16 - x^2 - y^2 = 0$ -----3)

Solve 1),2) and 3)

Case1 $X_1 = 2\sqrt{2}$, $y_1 = 2\sqrt{2}$ and $\lambda = \frac{1}{2}$

Case2 $X_2 = -2\sqrt{2}$, $y_2 = -2\sqrt{2}$ and $\lambda = \frac{1}{2}$

But, x,y >0 , so only case1 can be used.

2.<u>SOC:</u> $[H] = \begin{bmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 1 \\ 2y & 1 & -2\lambda \end{bmatrix}$

$X_1 = 2\sqrt{2}$, $y_1 = 2\sqrt{2}$ and $\lambda = \frac{1}{2}$

$|H_2| = \begin{vmatrix} 0 & 4\sqrt{2} & 4\sqrt{2} \\ 4\sqrt{2} & -1 & 1 \\ 4\sqrt{2} & 1 & -1 \end{vmatrix} = 256 > 0$ (Maximum)

So, at $X_1 = 2\sqrt{2}$, $y_1 = 2\sqrt{2}$gives Z maximum = 8

2. a) $U(x,y) = xy$ subject to $x+y=200$

$L(x,y,\lambda) = xy + \lambda(200 - x - y)$

1.<u>FOC:</u> $Lx = y + \lambda(-1) = 0$ -----1)

$Ly = x + \lambda(-1) = 0$ -----2)

$L\lambda = 200 - x - y = 0$ -----3)

Solve 1),2) and 3) x=100 , y=100 and λ= 100

2.<u>SOC:</u> $[H] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 > 0$ (Maximum)

So, at x=100 , y=100 gives U maximum = 10000 which is the equilibrium of the consumer with this constraint.

3. $TC = 8x^2 - xy + 12y^2$ subject to $x = 42 - y$

$L(x,y,\lambda) = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$

1.<u>FOC:</u> $Lx = 16x - y + \lambda(-1) = 0$ -----1)

$Ly = 24y - x + \lambda(-1) = 0$ -----2)

$L\lambda = 42 - x - y = 0$ -----3)

Solve 1),2) and 3) x=25 , y=17 and λ= 383

2.<u>SOC:</u> $[H] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 16 & -1 \\ 1 & -1 & 24 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 16 & -1 \\ 1 & -1 & 24 \end{vmatrix} = -42 < 0$ (Minimum)

So, at x=25 , y=17 gives minimal total cost = $8(25)^2 - (25)(17) + 12(17)^2$ = 8043, so the firm should produce x=25 and y=17 to minimize total cost.

4. U= XY+X subject to 6X+2Y=110

$L(x,y,\lambda) = XY + X + \lambda(110 - 6x - 2y)$

1.<u>FOC:</u> $Lx = y + 1 + \lambda(-6) = 0$ -----1)

$Ly = x + \lambda(-2) = 0$ -----2)

$L\lambda = 110 - 6x - 2y = 0$ -----3)

Solve 1),2) and 3) $x = \frac{28}{3}$, y=27 and $\lambda = \frac{14}{3}$

2.<u>SOC:</u> $[H] = \begin{bmatrix} 0 & 6 & 2 \\ 6 & 0 & 1 \\ 2 & -6 & -2 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 6 & 2 \\ 6 & 0 & 1 \\ 2 & -6 & -2 \end{vmatrix} = 12 > 0$ (Maximum)

So, at $x = \frac{28}{3}$, y=27 gives maximum utility = $(\frac{28}{3})(27) + \frac{28}{3} = 261$.

5. $TC = 2K + L$ subject to $120 = 4(KL)^{0.5}$

Given prices of capital and labor are 2 and 1 baht per unit, respectively, find the optimal level of K and L that a producer can minimize the production costs from producing 120 units

$L(K,L,\lambda) = 2K + L + \lambda(30 - K^{0.5}L^{0.5})$

1.<u>FOC:</u> $LK = 2 + \lambda(-0.5K^{-0.5}L^{0.5}) = 0$ -----1)

$LL = 1 + \lambda(-0.5K^{0.5}L^{-0.5}) = 0$ -----2)

$L\lambda = 30 - K^{0.5}L^{0.5} = 0$ -----3)

Solve 1),2) and 3) $K = 15\sqrt{2} = 21.21$, $L = 30\sqrt{2} = 42.43$ and λ= 2.83

2.SOC: $[H]=\begin{bmatrix} 0 & 0.5K^{-0.5}L^{0.5} & 0.5\,K^{0.5}L^{-0.5} \\ 0.5K^{-0.5}L^{0.5} & 0.25\,K^{-1.5}L^{0.5}\,\lambda & -0.25\,K^{-0.5}L^{-0.5}\,\lambda \\ 0.5\,K^{0.5}L^{-0.5} & -0.25\,K^{-0.5}L^{-0.5}\lambda & 0.25\,K^{0.5}L^{-1.5}\lambda \end{bmatrix}$

$|H_2| < 0$ (Minimum)

So, at K=21.21, L=42.43 gives minimal total cost.

6. U(x,y) = (x+2)(y+1)=xy+2y+x+2 subject to 4x+6y=130

L (x,y,λ) = xy+2y+x+2 + λ(130 – 4x –6y)

1.FOC: Lx= y+1+ λ(-4)=0 -----1)

Ly= x+2 + λ(-6)=0 -----2)

Lλ= 130 – 4x –6y =0 -----3)

Solve 1),2) and 3) x=16 , y=11 and λ= 3

2.SOC: $[H]=\begin{bmatrix} 0 & 4 & 6 \\ 4 & 0 & 1 \\ 6 & 1 & 0 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 4 & 6 \\ 4 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = 48 > 0$ (Maximum)

So, at x=16 , y=11 gives U maximum = (18)(12)= 216which is the equilibrium of the consumer with this constraint.

7. U=$\sqrt{xy}$ subject to 5x+4y=100

L (x,y,λ) = $\sqrt{xy}$ + λ(100 – 5x –4y)

1.FOC: Lx= $0.5y^{0.5}\,x^{-0.5}$+ λ(-5)=0 -----1)

Ly= $0.5y^{-0.5}\,x^{0.5}$+ λ(-4)=0 -----2)

Lλ= 100 – 5x –4y =0 -----3)

Solve 1),2) and 3) x=10 , y=12.5 and λ= 0.112

2.SOC: $[H]=\begin{bmatrix} 0 & 4 & 5 \\ 4 & -0.25y^{0.5}\,x^{-1.5} & 0.25y^{-0.5}\,x^{-0.5} \\ 5 & 0.25y^{-0.5}\,x^{-0.5} & -0.25y^{-1.5}\,x^{0.5} \end{bmatrix}$

$|H_2| > 0$ (Maximum)

So, at x=10 , y=12.5 gives U maximum =11.18.

8. $TC= 15+15Q_1+Q_1^2+5Q_2+4Q_2^2$ subject to $Q_1+Q_2 =100$

$L(Q_1,Q_2,\lambda) = 15+15Q_1+Q_1^2+5Q_2+4Q_2^2 + \lambda(100 - Q_1 - Q_2)$

1.FOC: $LQ_1= 15+ 2Q_1+ \lambda(-1)=0$ -----1)

$LQ_2= 5+ 8Q_2+\lambda(-1)=0$ -----2)

$L\lambda= 100 - Q_1 - Q_2=0$ -----3)

Solve 1),2) and 3) $Q_1 =79$, $Q_2=21$ and $\lambda= 173$

2.SOC: $[H]= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 8 \end{vmatrix} = -10 <0$ (Minimum)

So, they should produce $Q_1 =79$, $Q_2=21$ so that it gives minimal total cost of production.

9. $P=-2X^2 - 3Y^2 +30X +40Y$ subject to $10X+20Y=410$

$L(x,y,\lambda) = -2X^2 - 3Y^2 +30X +40Y + \lambda(410 - 10x - 20y)$

1.FOC: $Lx= -4x+30 + \lambda(-10)=0$ -----1)

$Ly= -6y+40 + \lambda(-20)=0$ -----2)

$L\lambda= 410 - 10x - 20y =0$ -----3)

Solve 1),2) and 3) $x=13$, $y=14$ and $\lambda=-2.2$

2.SOC: $[H]= \begin{bmatrix} 0 & 10 & 20 \\ 10 & -4 & 0 \\ 20 & 0 & -6 \end{bmatrix}$

$|H_2| = \begin{vmatrix} 0 & 10 & 20 \\ 10 & -4 & 0 \\ 20 & 0 & -6 \end{vmatrix} = 2200>0$ (Maximum)

So, they should produce $x=13$, $y=14$ to maximizes production per hour (P).

10. $X=72 - 0.5 P_x$, so $Px= 144 - 2X$, then $TR_x=144X - 2X^2$

$Y=120 - P_y$, so $Py=120 - Y$, then $TR_y =120Y - Y^2$

$TR= TR_x+TR_y= 144X - 2X^2 +120Y - Y^2$

$TC=X^2 +XY +Y^2 +35$

So, total profit = TR – TC $= 144X - 2X^2 + 120Y - Y^2 - [X^2 + XY + Y^2 + 35]$

$$= 144X - 3X^2 + 120Y - 2Y^2 - XY - 35$$

Subject to X+Y=40

$L(x,y,\lambda) = 144X - 3X^2 + 120Y - 2Y^2 - XY - 35 + \lambda(40 - x - y)$

1. <u>FOC:</u> $L_x = 144 - 6X - Y + \lambda(-1) = 0$ -----1)

   $L_y = 120 - 4Y - X + \lambda(-1) = 0$ -----2)

   $L_\lambda = 40 - x - y = 0$ -----3)

   Solve 1),2) and 3) x=18, y=22 and λ=14

2. <u>SOC:</u> $[H] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -6 & -1 \\ 1 & -1 & -4 \end{bmatrix}$

   $|H_2| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -6 & -1 \\ 1 & -1 & -4 \end{vmatrix} = 8 > 0$ (Maximum)

So, they should produce x=18, y=22 to maximizes total profit.