

Example 2.3. Let $f(x) = \frac{2}{\sqrt{x-1}}$ and $g(x) = x^2 + 3$. Find $g \circ f$ and $f \circ g$. Find $(g \circ f)(2)$.

$D_f = (1, \infty)$
 $R_f = (0, \infty)$
 $x-1 > 0 \Rightarrow x > 1$
 $\sqrt{x-1} > 0$

$D_g = \mathbb{R}$
 $R_g = [3, \infty)$
 $x^2 \geq 0$
 $x^2 + 3 \geq 3$

$g \circ f$: check $R_f \cap D_g \neq \emptyset \checkmark \Rightarrow R_f \cap D_g = (0, \infty) \neq \emptyset$
 Since $R_f \subseteq D_g$, then $D_{g \circ f} = D_f = (1, \infty)$.

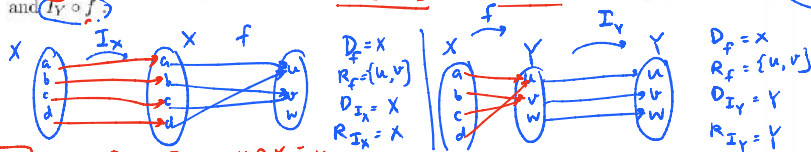
$(g \circ f)(x) = g(f(x)) = g\left(\frac{2}{\sqrt{x-1}}\right) = \left(\frac{2}{\sqrt{x-1}}\right)^2 + 3$
 $= \frac{4}{|x-1|} + 3$
 for $x \in D_{g \circ f} = (1, \infty)$

$f \circ g$: check: $R_g \cap D_f \neq \emptyset \checkmark$
 $(f \circ g)(x) = f(g(x)) = f(x^2 + 3) = \frac{2}{\sqrt{(x^2+3)-1}} = \frac{2}{\sqrt{x^2+2}}$
 $D_{f \circ g} = D_g = \mathbb{R}$ because $R_g \subseteq D_f$.

Example 2.4. (Composition with the Identity Function):
 Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f: X \rightarrow Y$ is given by

$f(a) = u, f(b) = v, f(c) = v, f(d) = u.$

Let $I_X: X \rightarrow X$ and $I_Y: Y \rightarrow Y$ with $I_X(x) = x$ and $I_Y(y) = y$ be identity functions. Find $f \circ I_X$ and $I_Y \circ f$.



- $f \circ I_X$: $R_{I_X} \cap D_f = X \cap X = X \neq \emptyset$
 $(f \circ I_X)(x) = f(I_X(x)) = f(x) \Rightarrow (f \circ I_X)(x) = f(x) \quad \forall x \in X.$
- $I_Y \circ f$: $R_f \cap D_{I_Y} = \{u, v\} \cap Y = \{u, v\} \neq \emptyset$
 $(I_Y \circ f)(x) = I_Y(f(x)) = f(x) \Rightarrow (I_Y \circ f)(x) = f(x) \quad \forall x \in X$

Note: $I_X \circ f$ and $f \circ I_Y$ cannot be defined, because

$R_f \cap D_{I_X} = X \cap Y = \emptyset$
 and $R_{I_Y} \cap D_f = Y \cap X = \emptyset.$

Theorem 2.1. Composition with an Identity Function

If f is a function from a set X to a set Y , and I_X is the identity function on X , and I_Y is the identity function on Y , i.e.,

$I_X(x) = x \quad \forall x \in X, \quad I_Y(y) = y \quad \forall y \in Y$

then

$f \circ I_X = f \quad \text{and} \quad I_Y \circ f = f.$

Let f be a function from a set X to a set Y , and suppose f has an inverse function f^{-1} . Recall that f^{-1} is the function from Y to X with the property that

$$f^{-1}(y) = x \iff f(x) = y.$$

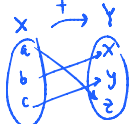
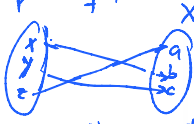
What happens when f is composed with f^{-1} ? Or when f^{-1} is composed with f ?

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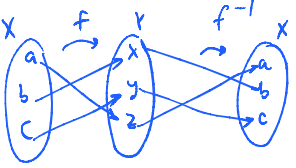
Example 2.5. (Exercise) Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Define $f : X \rightarrow Y$ by

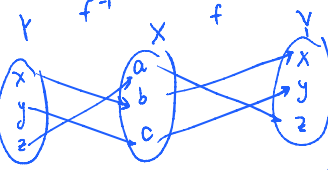
$$f(a) = z, \quad f(b) = x, \quad f(c) = y.$$

1. Draw the arrow diagram for f .
2. Show that the inverse function f^{-1} exists for the function f .
3. Find f^{-1} and draw the arrow diagram for f^{-1} .
4. Find and draw the arrow diagrams for $f^{-1} \circ f$ and $f \circ f^{-1}$. Compare with I_X and I_Y .

①  $f = \{(a, z), (b, x), (c, y)\}$ ③ 
 $\therefore f^{-1} = \{(z, a), (x, b), (y, c)\}$
 $f^{-1} : Y \rightarrow X$

- ② To show that the inverse function of f exists, we will show that f^{-1} is a function.
- ① From the arrow diagram of f^{-1} , we have that every element in Y (domain of f^{-1}) gets mapped.
 - ② Each element in Y (domain of f^{-1}) get mapped only once.
- Hence, from ① and ②, f^{-1} is a function and the inverse function of f exists.

④  $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(z) = a$
 $(f^{-1} \circ f)(b) = f^{-1}(f(b)) = f^{-1}(x) = b$
 $(f^{-1} \circ f)(c) = f^{-1}(f(c)) = f^{-1}(y) = c$
 $\therefore f^{-1} \circ f : X \rightarrow X$ is an identity function on X
 or $f^{-1} \circ f = I_X$

 $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(b) = x$
 $(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(c) = y$
 $(f \circ f^{-1})(z) = f(f^{-1}(z)) = f(a) = z$
 $\therefore f \circ f^{-1} : Y \rightarrow Y$ is an identity function on Y .
 or $f \circ f^{-1} = I_Y$.

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Theorem 2.2. Composition of a Function with Its Inverse

If $f: X \rightarrow Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \rightarrow X$, then

$$f^{-1} \circ f = I_X \quad \text{and} \quad f \circ f^{-1} = I_Y.$$

Example 2.6. (Exercise) Show that the following statement is true.

Theorem 2.3. The composition of two injective (one-to-one) functions is also injective. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both **one-to-one** functions, then $g \circ f$ is **one-to-one**.

Note: f is 1-1: For $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
 g is 1-1: For $y_1, y_2 \in Y$, if $g(y_1) = g(y_2)$, then $y_1 = y_2$.

consider $(g \circ f)(x) = g(f(x))$.

Let $x_1, x_2 \in X$. Suppose

Then

• Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$,

$$g(y_1) = g(y_2) \Rightarrow y_1 = y_2 \quad \text{or} \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

That is, $(g \circ f)(x_1) = (g \circ f)(x_2)$ implies $x_1 = x_2$ and hence, $g \circ f$ is one-to-one. #

Example 2.7. (Exercise) Show that the following statement is true.

Theorem 2.4. The composition of two surjective (onto) functions is also surjective (onto). If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both **onto** functions, then $g \circ f$ is **onto**.

Note: f is onto: Let $y \in Y$. $\exists x \in X$, $f(x) = y$.
 g is onto: Let $z \in Z$. $\exists y \in Y$, $g(y) = z$.

Since $f: X \rightarrow Y$, $g: Y \rightarrow Z$, then $g \circ f$ is a function from X to Z .

Let $z \in Z$. Then $\exists y \in Y$ such that $g(y) = z$ (since g is onto).
 Since $y \in Y$ (codomain of f), $\exists x \in X$, such that $f(x) = y$.

I.e. Given $z \in Z$ (codomain of $g \circ f$), there exists $x \in X$ such that $z = g(y) = g(f(x)) = (g \circ f)(x)$

Hence, $g \circ f$ is onto. #