

Curve Sketching

TU152: Fundamental Mathematics

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Curve Sketching

Let f be a relation from X to Y where $X, Y \subseteq \mathbb{R}$.

Recall that we can write f as

$$f = \{(x, y) | x \in X, y \in Y\}$$

or we can write $y = f(x)$, where $x \in X$. We are interested in how to sketch a graph or curve of a give relation or function f .

Definition (Graph/Curve)

Let X and Y be subsets of $\mathbb{R}$. For a relation $f : X \rightarrow Y$, each ordered pair $(x, f(x))$, $x \in X$, can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph or curve of f** .

Curve Sketching

Let f be a relation from X to Y where $X, Y \subseteq \mathbb{R}$. To sketch a curve for f , it is helpful to identify the following characteristics of the curve.

- (1) Domain and Range (undefined points)
- (2) Intercepts: the x -and y -intercepts
- (3) Symmetry: about the x -axis/ y -axis/origin
- (4) Asymptotes: Vertical/Horizontal/Slant Asymptotes
- (5) Increasing/Decreasing Intervals
- (6) Local (Relative) Maximum/Minimum
- (7) Concavity and Points of Inflection

By using the information obtained from steps (1) to (7)., we can sketch the curve as follows.

- First, draw dashed lines for the asymptotes of the function.
- Then plot the x -and y -intercepts, maximum and minimum points and points of inflection on the graph.
- Sketch the curve between the points, using the intervals of increase and decrease and intervals of concavity.
- Be sure that the graph behaves correctly when approaching asymptotes.

Remarks

- The steps (1)-(3) can be done by using some basic algebra knowledge.
- For steps (4)-(7), the basic calculus knowledge would be very helpful to identify these characteristics.

Curve Sketching for $f : X \rightarrow Y, X, Y \subseteq \mathbb{R}$

- (1) **Domain & Range** of f : Find the set of all values x such that f is well-defined and the corresponding y . This will be useful when finding vertical asymptotes and determining critical numbers.
- (2) **Intercepts**: Find the x - and y -intercepts for $(x, y) \in f$, if possible.
 - To find the x -intercept, set $y = 0$ and solve the equation for x .
 - To find the y -intercept, set $x = 0$ and solve the equation for y .
- (3) **Symmetry**: The curve is...

• **Symmetric about the x -axis** if $(x, y) \in f$ implies $(x, -y) \in f$.

• **Symmetric about the y -axis** if $(x, y) \in f$ implies $(-x, y) \in f$.

When f is a function, denoted by $y = f(x)$, the condition above becomes

$$f(x) = f(-x), \quad \forall x \in X$$

and the function f is called “even.”

• **Symmetric about the origin** if $(x, y) \in f$ implies $(-x, -y) \in f$.

When f is a function, denoted by $y = f(x)$, the condition above becomes

$$f(-x) = -f(x), \quad \forall x \in X$$

and the function f is called “odd.”

Example Determine the symmetry of the graph for each of the following relations.

① $f_1 = \{(x, y) | y = x^2\}$

② $f_2 = \{(x, y) | x = y^2\}$

③ $f_3 = \{(x, y) | y = -x\}$

④ $f_4 = \{(x, y) | x^2 + y^2 = 4\}$

- (4) Asymptotes of a curve can be determined in many different ways. The followings are common two approaches: (i) re-arranging the term and (ii) using the limit.

(i) **Re-arranging the term:** Consider the relation of the form

$$F(x, y) = C$$

such that $F(x, y)$ can be factored into linear terms, e.g. $x - a, y - b, mx + ny$, and C is a nonzero constant.

- **The vertical asymptote:** line $x = a$
- **The horizontal asymptote:** line $y = b$
- **The slant asymptote:** line $mx + ny = 0$

E.g.

- To obtain a **vertical asymptote** (e.g. line $x = a$), arrange the equation in the form $y = f(x)$ and find a such that the *denominator* of $f(a)$ is zero.
Note: In this case, a is not in the domain of f .
- To obtain a **horizontal asymptote** (e.g. line $y = b$), arrange the equation in the form $x = g(y)$ and find b such that the *denominator* of $g(b)$ is zero.
Note: In this case, b is not in the range of f .

(ii) **Using the limit:** This method is often used when the relation f is a function $f : X \rightarrow Y, y = f(x), x \in X$.

- **Vertical asymptote:** By using the equation in the form $y = f(x)$,

$x = a$ is a vertical asymptote if

$$\begin{array}{l} \lim_{x \rightarrow a} f(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \infty \quad \text{or} \\ \lim_{x \rightarrow a} f(x) = -\infty \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = -\infty. \end{array}$$

Alternatively, if it is possible to re-arrange $x = g(y)$, $x = a$ is a vertical asymptote when $\lim_{y \rightarrow \infty} g(y) = a$ or $\lim_{y \rightarrow -\infty} g(y) = a$.

- **Horizontal asymptote:** By using the equation in the form $y = f(x)$,

$y = b$ is a horizontal asymptote if

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

Alternatively, if it is possible to re-arrange $x = g(y)$, $y = b$ is a horizontal asymptote when

$$\begin{array}{l} \lim_{y \rightarrow b} g(y) = \infty \quad \text{or} \quad \lim_{y \rightarrow b^+} g(y) = \infty \quad \text{or} \quad \lim_{y \rightarrow b^-} g(y) = \infty \quad \text{or} \\ \lim_{y \rightarrow b} g(y) = -\infty \quad \text{or} \quad \lim_{y \rightarrow b^+} g(y) = -\infty \quad \text{or} \quad \lim_{y \rightarrow b^-} g(y) = -\infty. \end{array}$$

- **Slant asymptote:** By using the equation in the form $y = f(x)$, the line

$y = mx + b$ is a slant asymptote if

$$\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} [f(x) - (mx + b)] = 0.$$

Example: Consider the relation defined by

$$f = \{(x, y) | x^2 + xy - x - y - 3 = 0\}.$$

Find all asymptotes for the curve of f .

Example: Consider a function

$$f(x) = \frac{x^2 + 5x + 4}{x^2}.$$

- 1 Find domain and range of f .
- 2 Find x -intercepts and y -intercepts.
- 3 Determine the symmetry of f .
- 4 Find the asymptotes for f (if any).

Example(continued):

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The characteristics (5)-(7), increasing/decreasing, extrema, and concavity, of the curves of a relation $f : X \rightarrow Y$ can generally be identified when f is a *continuous* and *differentiable* function on a subset of its domain $x \subseteq \mathbb{R}$ by using some basic *calculus*. The followings are some calculus required to used for identifying these characteristics.

Definition

A critical number of a function $f : X \rightarrow Y$ is a number c in its domain X for which

$$f'(c) = 0, \quad \text{or} \quad f'(c) \text{ does not exists.}$$

Example: Find the domain and the critical numbers for each of the following functions.

- $f(x) = (x + 4)^{2/3}$
- $f(x) = \frac{x^2}{x-1}$.

(5) **Increasing/Decreasing Intervals:** (defined below) can be identified by using the first derivative from the following theorem.

Definition (Increasing/Decreasing Function)

Let $f : X \rightarrow Y$ be a function $y = f(x)$, $X, Y \subseteq \mathbb{R}$. Let $S \subseteq X$.

- f is increasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is decreasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Theorem: Test for Increasing/Decreasing

Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b) .

- If $f'(x) > 0$ for all x in (a, b) , then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b) , then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for all x in $[a, b]$, then f is a constant on the interval.

Let $f : X \rightarrow Y$, $X, Y \subset \mathbb{R}$ be a function.

- (6) **Extrema:** (defined below) of f can be identified by using the first derivatives and the second derivatives from the following theorems.

Definition (Relative Extrema: Maximum/Minimum)

- A number $f(c)$ is a **relative maximum** of a function f if $f(x) \leq f(c)$ for every x in *some* open interval that contains c .
- A number $f(c)$ is a **relative minimum** of a function f if $f(x) \geq f(c)$ for every x in *some* open interval that contains c .

Theorem: Relative Extrema Occur at Critical numbers

If a function f has a relative extremum at $x = c$, then c is a critical number.

From the previous theorem, we can find an extremum by first find all critical numbers c of f first. Then, we can use two following approaches to identify if each critical number gives maximum, minimum, or it does not give any extremum.

- The First Derivative Test for Relative Extrema
- The Second Derivative Test for Relative Extrema

Theorem: First Derivative Test for Relative Extrema

Let f be continuous on $[a, b]$ and differentiable on (a, b) except possibly at the critical number c .

- (i) If $f'(x)$ changes from positive to negative at c , then $f(c)$ is a relative **maximum**.
- (ii) If $f'(x)$ changes from negative to positive at c , then $f(c)$ is a relative **minimum**.
- (iii) If $f'(x)$ has the same algebraic sign on each side of c , then $f(c)$ is **not an extremum**.

Theorem: Second Derivative Test for Relative Extrema

Let f be a function for which the second derivative f'' exists on an interval (a, b) that contains the critical number c .

- (i) If $f''(c) > 0$, then $f(c)$ is a relative **minimum**.
- (ii) If $f''(c) < 0$, then $f(c)$ is a relative **maximum**.
- (iii) If $f''(c) = 0$, the test fails and $f(c)$ may or may not be a relative extremum. In this case, use the First derivative Test.

Notice that, to find an extremum, we have to first find all critical numbers c of f .

Example: Consider the function $f(x) = x^3 - 3x^2 - 9x + 2$

- 1 Find domain of f .
- 2 Find x -intercepts and y -intercepts (if any).
- 3 Determine the symmetry of f . Determine whether f is even or odd.
- 4 Find the asymptotes for f (if any).
- 5 Find the critical number of f .
- 6 Determine the intervals on which f is increasing and decreasing.
- 7 Determine the extrema (maximum and minimum) of f by using
 - the First Derivative Test,
 - the Second Derivative test.

Example(continued):

Example(continued):

Let $f : X \rightarrow Y$ be a function with $X, Y \subseteq \mathbb{R}$.

- (7) **Concavity and the point of inflection:** (defined below) of f can be identified by using the the second derivatives from the following theorem.

Definition (Concavity: concave up/concave down)

Let f be a differentiable function on an interval (a, b) .

- (i) If f' is an increasing function on (a, b) , then the graph of f is **concave up** on the interval.
- (ii) If f' is a decreasing function on (a, b) , then the graph of f is **concave down** on the interval.

Definition (Point of Inflection)

Let f be a continuous on an interval (a, b) containing the number c . A point $(c, f(c))$ is a **point of inflection** of the graph f if there is a tangent line at $(c, f(c))$ and the graph changes concavity at this point.

Theorem: Test for Concavity

Let f be function for which f'' exists on (a, b) .

- (i) If $f''(x) > 0$ for all x in (a, b) , then the graph of f is concave up on (a, b) .
- (ii) If $f''(x) < 0$ for all x in (a, b) , then the graph of f is concave down on (a, b) .

Theorem: Point of Inflection

If $(c, f(c))$ is a point of inflection for the graph of a function f , then either

$$f''(c) = 0 \quad \text{or} \quad f''(c) \text{ does not exist.}$$

Example: Consider the function $f(x) = 4x^4 - 4x^2$.

- 1 Find domain of f .
- 2 Find x -intercepts and y -intercepts (if any).
- 3 Determine the symmetry of f . Determine whether f is even or odd.
- 4 Find the asymptotes for f (if any).
- 5 Find the critical numbers of f .
- 6 Determine the intervals on which f is increasing and decreasing.
- 7 Determine the relative extrema (maximum and minimum) of f .
- 8 Use the information above to sketch the graph of f .

Example(continued):

Example(continued):

Example: