Vector Autoregressive (VARs) models

Interdependent and Dynamic System Models

Process of using VARs models

- Set of variables
- Optimal number of lags
- Stability Test – Unit Roots test
- Exogeneity Test – Granger Test
- Evaluation of the model
- Implications of the model
  - Impulse Response Function
  - Forecast Error Variance Decomposition

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Model Specification

Consider simple bivariate system:

$$x_t = b_{10} - b_{12} y_t + \gamma_{11} x_{t-1} + \gamma_{12} y_{t-1} + \varepsilon_{xt}$$
$$y_t = b_{20} - b_{21} x_t + \gamma_{21} x_{t-1} + \gamma_{22} y_{t-1} + \varepsilon_{yt}$$

Assume y_t and z_t are stationary.

ε_{xt} *and* ε_{yt} are uncorrelated white-noise disturbances.

Both x_t and y_t have contemporaneous effect on each other, thus, endogeneity problem occurs.

Model Specification

Structural-form can be rewritten in matrix:

$$\begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix} + \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}\begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{xt} \\ \varepsilon_{yt} \end{pmatrix}$$

or

$$BY_t = \Gamma_0 + \Gamma_1 Y_{t-1} + \boldsymbol{\varepsilon}_t$$

Reduced-form model can be derived by multiply B^{-1} and obtain VAR model in standard form:

$$Y_t = A_0 + A_1 Y_{t-1} + \in_t$$

Model Specification

VAR bivariate system in standard form:

$$Y_t = A_0 + A_1 Y_{t-1} + \in_t$$

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}$$

$$x_t = a_{10} + a_{11} x_{t-1} + a_{12} y_{t-1} + e_{1t}$$

$$y_t = a_{20} + a_{21} x_{t-1} + a_{22} y_{t-1} + e_{2t}$$

where

$$\begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix} \sim IID\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \right)$$

Stability

From $$Y_t = A_0 + A_1 Y_{t-1} + \in_t$$

$$Y_t = A_0 + A_1\left(A_0 + A_1 Y_{t-2} + \in_{t-1}\right) + \in_t$$

$$= \left(I + A_1\right) A_0 + A_1^2 Y_{t-2} + A_1 \in_{t-1} + \in_t$$

For n iterations

$$Y_t = \left(I + A_1 + \cdots + A_1^n\right) A_0 + \sum_{i=0}^{n} A_1^i \in_{t-i} + A_1^{n+1} Y_{t-n-1}$$

For stability property, convergence is required.

$$Y_t = \mu + \sum_{i=0}^{\infty} A_1^i \in_{t-i} \qquad where \quad \mu = \left[\bar{x} \quad \bar{y}\right]'$$

Stationarity

For Stationary property, means, variances, and covariances are needed to be constant.

$$\bar{x} = \left[a_{10}\left(1 - a_{22}\right) + a_{12}a_{20} \right] / \Delta$$

$$\bar{y} = \left[a_{20}\left(1 - a_{11}\right) + a_{21}a_{10} \right] / \Delta$$

$$where \quad \Delta = \left(1 - a_{11}\right)\left(1 - a_{22}\right) - a_{12}a_{21}$$

Variance-Covariance Matrix:

$$E\left(Y_t - \mu\right)^2 = E\left[\sum_{i=0}^{\infty} A_1^i \in_{t-i} \right]^2$$

$$= \left(I + A_1^2 + A_1^4 + A_1^6 + \cdots\right)\Sigma$$

$$= \left(I - A_1^2\right)^{-1}\Sigma$$

Model Selection

In order to determine the lag order of VAR(p), AIC, SIC, & HQIC can be used:

$$AIC(p) = \log\left(\det\left(\hat{\Omega}_p\right)\right) + 2\frac{pm^2}{n}$$

$$SIC(p) = \log\left(\det\left(\hat{\Omega}_p\right)\right) + \log(n)\frac{pm^2}{n}$$

$$HQIC(p) = \log\left(\det\left(\hat{\Omega}_p\right)\right) + 2\log\left(\log(n)\right)\frac{pm^2}{n}$$

Exogeneity Test

Granger exogeneity test

$$\begin{pmatrix} A_{11}(L) & A_{12}(L) \\ A_{21}(L) & A_{22}(L) \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

To test whether X_t is exogenous variables, the hypotheses is to test

$$A_{21}(L) = 0 \ and \ A_{12}(L) = 0$$

Impulse Response Function

From
$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Stability property:
$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \sum_{i=0}^{\infty} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^i \begin{pmatrix} \varepsilon_{1t-i} \\ \varepsilon_{2t-i} \end{pmatrix}$$

Derive to be impulse response function
$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \sum_{i=0}^{\infty} \begin{pmatrix} \phi_{11}(i) & \phi_{12}(i) \\ \phi_{21}(i) & \phi_{22}(i) \end{pmatrix} \begin{pmatrix} \varepsilon_{yt-i} \\ \varepsilon_{zt-i} \end{pmatrix}$$

where $\phi_{11}(i), \phi_{12}(i), \phi_{21}(i), \phi_{22}(i)$ *are impulse response functions*

Impulse Response Function

<u>Example</u>: $$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Suppose $$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.5 \end{pmatrix}$$

and Var-Cov $$\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 16 & 14 \\ 14 & 25 \end{pmatrix}$$

Set $$Y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ and $$\in_1 = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Then, $$Y_1 = AY_0 + \in_1 = \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Impulse Response Function

Example:

Then, $$Y_2 = AY_1 + \in_2 = \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.5 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.6 \\ 0.8 \end{pmatrix}$$

and $$Y_3 = AY_2 + \in_3 = \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.5 \end{pmatrix}\begin{pmatrix} 1.6 \\ 0.8 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.72 \\ 0.72 \end{pmatrix}$$

Impulse of x on

Period	x	y
1	4.00	0.00
2	1.60	0.80
3	0.72	0.72
4	0.36	0.50
5	0.19	0.32

Impulse of y on

Period	x	y
1	0.00	5.00
2	0.50	2.50
3	0.45	1.35
4	0.32	0.77

Orthogonal Innovation

In general, the innovations in VAR are not contemporaneously independent of one another.

Thus, set of orthogonal innovations is used to analyze impulse response function analysis.

$$\mathbf{u}_t = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$ where: $$u_{1t} = b_{11}\varepsilon_{1t} \qquad b_{11} = \frac{1}{s_1}$$ and $$u^*_{2t} = \varepsilon_{2t} - b_{21}\varepsilon_{1t} \qquad u_{2t} = \frac{u^*_{2t}}{s_{2.1}}$$

s_1 is standard deviation of ε_{1t} and $s_{2.1}$ is standard error of regression u^*_{2t}

Orthogonal Innovation

Transformation of innovation matrix to be orthogonal innovations can be stated as:

$$\mathbf{u}_t = P \in_t \quad \text{or} \quad \in_t = P^{-1}\mathbf{u}_t$$

where: $P = \begin{pmatrix} \dfrac{1}{s_1} & 0 \\ \dfrac{-b_{21}}{s_{2.1}} & \dfrac{1}{s_{2.1}} \end{pmatrix}$ then, $P^{-1} = \begin{pmatrix} s_1 & 0 \\ b_{21}s_1 & s_{2.1} \end{pmatrix}$

Sample covariance matrix for $\mathbf{u}_t$ is:

$$\frac{1}{n}\sum \mathbf{u}_t\mathbf{u}_t' = P\left(\frac{1}{n}\sum \in_t \in_t'\right)P' = P\hat{\Omega}P' = I$$

Then, Choleski Factorization of matrix $\hat{\Omega}$ is:

$$\hat{\Omega} = P^{-1}\left(P^{-1}\right)'$$

Choleski Factorization

In analyzing VARs, identification of VARs system using Choleski factorization can be applied by setting order for contemporaneous effects of each variables.

Theoretical framework should be used for setting the order.

However, Granger Causality test can sometime be used in testing for the appropriated order.

IRF – Orthogonal Innovation

From previous example

$$s_1 = \sqrt{16} = 4 \qquad b_{21} = 14/16 = 0.875$$

$$s_{2.1} = \sqrt{s_2^2\left(1 - r_{12}^2\right)} = \sqrt{25\left[1 - \left((14)^2 / (16)(25)\right)\right]} = 3.5707$$

Thus, $P^{-1} = \begin{pmatrix} 4 & 0 \\ 3.5 & 3.5707 \end{pmatrix}$

Let $\mathbf{u}_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Then, $\in_1 = P^{-1}\mathbf{u}_1 = \begin{pmatrix} 4 & 0 \\ 3.5 & 3.5707 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3.5 \end{pmatrix}$

IRF – Orthogonal Innovation

From previous example

Impulse Responses from $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Period	x	y
1	4.00	3.50
2	1.95	2.55
3	1.04	1.67
4	0.58	1.04

Forecast Error Variance Decomposition

Forecast error variance decomposition reveals the proportion of the movements in a sequence due to its own shocks versus shocks to the other variable.

However, since variance decomposition also contains identification problem, the Choleski factorization is also applied by setting up the order of variables.

Forecast Error Variance Decomposition

Forecast Error

From VAR(1): $Y_t = AY_{t-1} + \in_t$

1-period forecast $\hat{Y}_{t+1} = E\left(Y_{t+1} \middle| Y_t, \ldots, Y_1\right) = AY_t$

2-period forecast $Y_{t+2} = A^2 Y_t + A \in_{t+1} + \in_{t+2}$

Then $\hat{Y}_{t+2} = A^2 Y_t$

In general: $Y_{t+s} = A^s Y_t + A^{s-1} \in_{t+1} + \cdots + A \in_{t+s-1} + \in_{t+s}$

Then $\hat{Y}_{t+s} = A^s Y_t$

Forecast Error:

$$Y_{t+s} - \hat{Y}_{t+s} = \in_{t+s} + A \in_{t+s-1} + \cdots + A^{s-1} \in_{t+1}$$

Forecast Error Variance Decomposition

Forecast Error

Var-Cov matrix for forecast errors s periods ahead.

$$\sum(s) = \Omega + A\Omega A' + A^2\Omega(A')^2 + \cdots + A^{s-1}\mathrm{A}\Omega(A')^{s-1}$$

From Choleski factorization $\hat{\Omega} = P^{-1}(P^{-1})'$

Then,

$$\sum(s) = P^{-1}(P^{-1})' + (AP^{-1})(AP^{-1})' + \cdots + (A^{s-1}P^{-1})(A^{s-1}P^{-1})'$$

Forecast Error Variance Decomposition

Variance Decomposition

1-period ahead forecast error variance:

$$\operatorname{var}(\in_t) = \Omega$$

Then, $\Omega = P^{-1}\operatorname{var}(\mathbf{u}_t)\left(P^{-1}\right)'$

$$= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11}v_1 & c_{12}v_2 \\ c_{21}v_1 & c_{22}v_2 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11}^2 v_1 + c_{12}^2 v_2 & c_{11}c_{21}v_1 + c_{12}c_{22}v_2 \\ c_{11}c_{21}v_1 + c_{12}c_{22}v_2 & c_{21}^2 v_1 + c_{22}^2 v_2 \end{pmatrix}$$

Forecast Error Variance Decomposition

Variance Decomposition

According to Orthogonal innovations,

$\mathrm{var}(u_1) = v_1 = 1$ & $\mathrm{var}(u_2) = v_2 = 1$ **and** $c_{12} = 0$

Then, $$\Omega = \begin{pmatrix} c_{11}^2 v_1 + c_{12}^2 v_2 & c_{11}c_{21}v_1 + c_{12}c_{22}v_2 \\ c_{11}c_{21}v_1 + c_{12}c_{22}v_2 & c_{21}^2 v_1 + c_{22}^2 v_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_{11}^2 v_1 & c_{11}c_{21}v_1 \\ c_{11}c_{21}v_1 & c_{21}^2 v_1 + c_{22}^2 v_2 \end{pmatrix} = \begin{pmatrix} c_{11}^2 & c_{11}c_{21} \\ c_{11}c_{21} & c_{21}^2 + c_{22}^2 \end{pmatrix}$$

Variances of 1-period forecast error are:

$\mathrm{var}\left(\hat{x}_1\right) = c_{11}^2$ **and** $\mathrm{var}\left(\hat{y}_1\right) = c_{21}^2 + c_{22}^2$

Forecast Error Variance Decomposition

Variance Decomposition

Variance of $\hat{x}_1$ can be decomposed as 1 component c_{11}^2 caused by innovation from x.

Variance of $\hat{y}_1$ can be decomposed as 2 components c_{21}^2 and c_{22}^2, where,

$\dfrac{c_{21}^2}{c_{21}^2 + c_{22}^2}$ represents % of variance of $\hat{y}_1$ caused by orthogonal innovation from x.

$\dfrac{c_{22}^2}{c_{21}^2 + c_{22}^2}$ represents % of variance of $\hat{y}_1$ caused by orthogonal innovation from y.

FEVD

From previous example: $A = \begin{pmatrix} 0.4 & 0.1 \\ 0.2 & 0.5 \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3.5 & 3.5707 \end{pmatrix} \quad P^{-1}\left(P^{-1}\right)' = \Omega = \begin{pmatrix} 16 & 14 \\ 14 & 25 \end{pmatrix}$$

$$AP^{-1} = \begin{pmatrix} 1.95 & 0.3571 \\ 2.55 & 1.7854 \end{pmatrix} \quad AP^{-1}\left(AP^{-1}\right)' = \begin{pmatrix} 3.93 & 5.61 \\ 5.61 & 9.69 \end{pmatrix}$$

$$A^2P^{-1} = \begin{pmatrix} 1.035 & 0.3214 \\ 1.665 & 0.9641 \end{pmatrix} \quad A^2P^{-1}\left(A^2P^{-1}\right)' = \begin{pmatrix} 1.1745 & 2.0331 \\ 2.0331 & 3.7017 \end{pmatrix}$$

FEVD of x on $\mathbf{u}_1$

Period	x	y
1	$\frac{4^2}{16} =$ 100.00%	$\frac{0}{16} =$ 0.00%
2	$\frac{\left(4^2+(1.95)^2\right)}{(16+3.93)} =$ 99.36%	$\frac{\left(0.3571^2\right)}{(16+3.93)} =$ 0.64%
3	$\frac{\left(4^2+(1.95)^2+(1.035)^2\right)}{(16+3.93+1.1745)} =$ 98.91%	$\frac{\left((0.3571)^2+(0.3214)^2\right)}{(16+3.93+1.1745)} =$ 1.09%