# CHAPTER 8

Optimization without Constraint: More-Than-One Independent Variable Cases

- Conditions for maximum or minimum
- Third degree price discrimination
- Competitive Firm Input Choices: Cobb-Douglas Technology
- Multiproduct-firm
- Multiplant-firm

## Optimization Condition

### One choice variable

The objective function: $z = f(x)$

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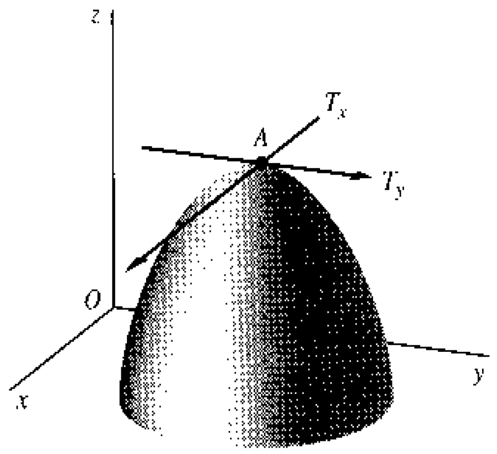
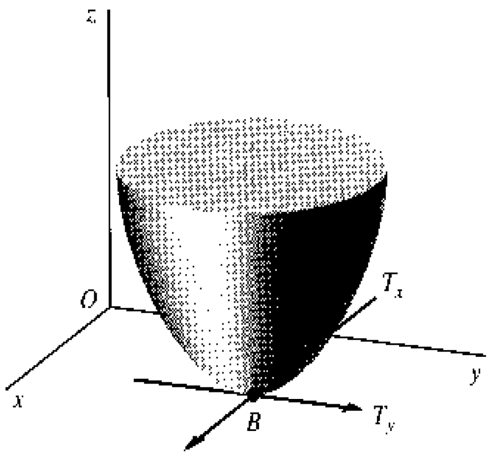
| Condition | Maximum | Minimum |
|---|---|---|
| First-order necessary<br>(FONC, FOC) | Derivative<br><br>Total differential | Derivative<br><br>Total differential |
| Second-order sufficient<br>(SOSC, SOC) | Derivative<br><br>Total differential | Derivative<br><br>Total differential |

**Extreme Value of a Function of Two Variables**

$z = f(x, y)$




**<u>First-Order Necessary Condition</u>**

$dz = 0$ for every $dx$ and $dy$

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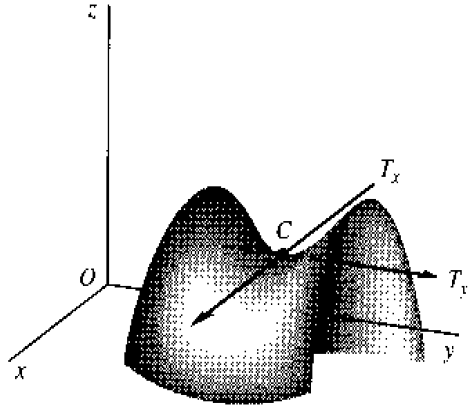
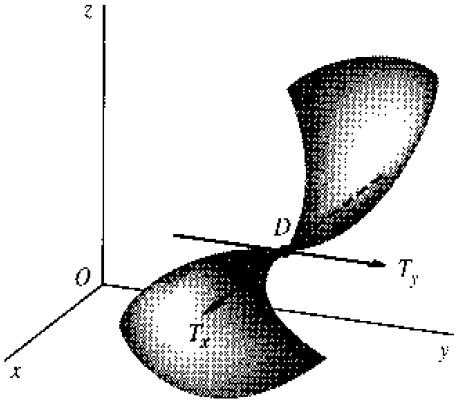
The first order necessary condition is necessary, but not sufficient to establish a maximum or a minimum.

**Saddle point**




**Inflection point**



**Second order sufficient condition**

The satisfaction of the first-order condition earmarks certain values of $z$ as the stationary values of the objective function. If at a stationary value of $z$ we find that $d^2z$ is positive definite, i.e., $d^2z > 0$, this will suffice to establish that value of $z$ as a minimum. Analogously, the negative definiteness of $d^2z$, $d^2z < 0$, is a sufficient condition for the stationary value to be a maximum.

**Second-order total differential, Hessian Matrix and Definiteness of Hessian Matrix**

🕮 **About "$H_k$"**

**Definition :** Let $A$ be an $n \times n$ matrix. A $k \times k$ submatrix, deriving from deleting the last $n - k$ columns and the last $n - k$ rows of matrix $A$ , can be called the $k^{th}$ order leading principal submatrix of matrix $A$. The corresponding determinant of this $k \times k$ submatrix is called the $k^{th}$ order leading principal minor of matrix $A$.

The $k^{th}$ order leading principal submatrix of matrix $A$ is denoted by $A_k$.
The $k^{th}$ order leading principal minor of matrix $A$ is denoted by $|A_k|$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Leading principal submatrices are:

$$[a_{11}]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Leading principal minors are:

$$|a_{11}|$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

For Hessian Matrix of function $y = f(x_1, x_2, \dots, x_n)$

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{bmatrix}$$

$$\_\_\_\_ = [f_{11}]$$

$$\_\_\_\_ = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$\_\_\_\_ = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

Leading principal minors are:

$$\_\_\_\_ = |f_{11}|$$

$$\_\_\_\_ = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$\_\_\_\_ = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

### 🕮 About "Definiteness of *H*"

Let ***H*** be a symmetric matrix with dimension $n \times n$

(a) ***H*** is **positive definite** if all n leading principal minors are positive

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, \ldots$$

(b) ***H* is negative definite if** n leading principal minor duly alternate in sign with the first one being negative.

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, \ldots$$

If (a) or (b) is not met, $H$ is indefinite.

Example: Find definiteness of matrix $B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

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Homework: Find definiteness of the following symmetric matrices ต่อไปนี้

(a) $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

**Second-Order Sufficient Condition (SOSC, SOC)**

For the maximum of objective function $z = f(x, y)$

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For the minimum of objective function $z = f(x, y)$

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Example: Find the maximum/minimum of function $z = x + 2ey - e^x - e^{2y}$

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## Objective functions with more three choice variables

$$z = f(x_1, x_2, x_3)$$

F.O.N.C :
$dz = 0$

$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0; dx_1, dx_2, dx_3 > 0$

$f_1 = f_2 = f_3 = 0$

We can use $f_1 = f_2 = f_3 = 0$ to find the stationary values/critical values

S.O.S.C

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

All leading principal minor
$|H_1| = |f_{11}|$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

$z$ is at the $\begin{bmatrix} \text{maximum} \\ \text{minimum} \end{bmatrix}$ if $\begin{cases} d^2z < 0 \\ d^2z > 0 \end{cases}$ if $\begin{cases} |H_1| < 0, |H_2| > 0, |H_3| < 0 \\ |H_1| > 0, |H_2| > 0, |H_3| > 0 \end{cases}$ if $\begin{cases} H \text{ is negative definite} \\ H \text{ is positive definite} \end{cases}$

**Objective functions with more $n$ −choice variables**

$$z = f(x_1, x_2, x_3, \dots, x_n)$$

F.O.N.C :

$$dz = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n = 0$$

$$\frac{\partial^2 f_i}{\partial x_i} = 0, for\ all\ i, i = 1,2, \dots n$$

S.O.S.C:

$$\therefore\ |H| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

All principal minors $|H_1|$ , $|H_2|$ ,… $|H_n|$

If $z\ is\ at$ is the maximum when $d^2z < 0$, $H$ is negative definite:

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, |H_4| > 0, |H_5| < 0, \dots$$

If $z\ is\ at$ is the minimum when $d^2z > 0$, $H$ is positive definite:

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, |H_4| > 0, |H_5| > 0, \dots$$

**Homework**:Find extreme values of the following functions and check for the sufficient condition

$z = 8x^3 + 2xy - 3x^2 + y^2 + 1$

$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$

$z = -3x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$

$z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$

$z = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$

$z = e^x + e^y + e^{w^2} - 2e^w - (x + y)$

$z = x^4 + x^2 - 6xy + 3y^2$

$w = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

$w = (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$

## Summary of optimal conditions

The objective function: $z = f(x, y)$

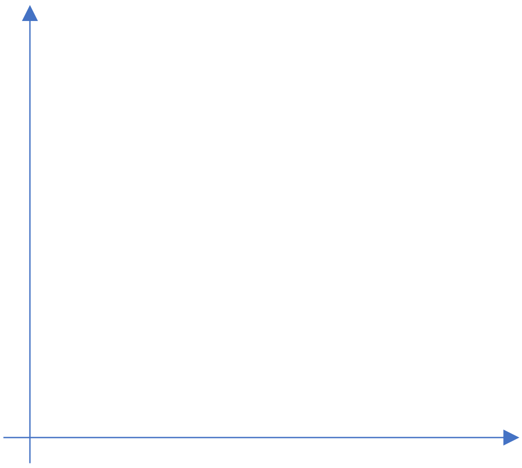
| Condition | Maximum | Minimum |
|---|---|---|
| First-order necessary (FONC, FOC) | $f_x = f_y = 0$ | $f_x = f_y = 0$ |
| Second-order sufficient (SOSC, SOC) | $\|H_1\| = f_{xx} < 0$<br>$\|H_2\| = f_{xx}f_{yy} - f_{xy}^2 > 0$ | $\|H_1\| = f_{xx} > 0$<br>$\|H_2\| = f_{xx}f_{yy} - f_{xy}^2 > 0$ |

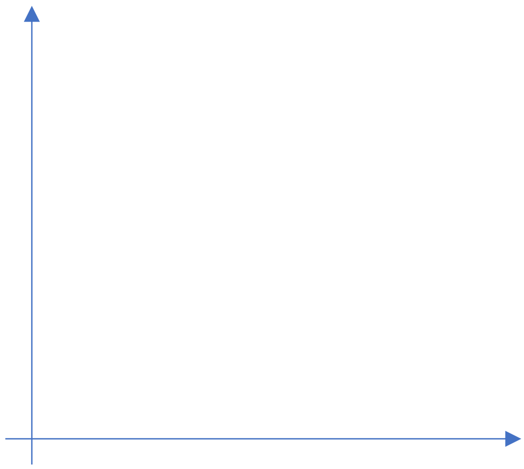
The objective function: $z = f(x_1, x_2, \dots, x_n)$

| Condition | Maximum | Minimum |
|---|---|---|
| First-order necessary (FONC, FOC) | $f_1 = f_2 = \cdots = f_n = 0$ | $f_1 = f_2 = \cdots = f_n = 0$ |
| Second-order sufficient (SOSC, SOC) | $H$ is negative definite.<br>$(-1)^k\|H_k\| > 0$<br>$k = 1,2, \dots, n$ | $H$ is positive definite.<br>$\|H_k\| > 0$<br>$k = 1,2, \dots, n$ |

**Third- Degree Price Discrimination**

Third-Degree Price Discrimination involves charging different prices to different segments of the market for the same good. This is to maximize profit within each segment of the market. For example, a theater may divide cinema goers into seniors, adults, and children, each paying a different price when seeing the same movie.





Total Revenue of a company is:

$$TR = R_1(Q_1) + R_2(Q_2) + R_3(Q_3),$$

$R_i$ is a total revenue function of market $i$.

Total cost of the company is:

$$C = C(Q)$$

$$Q = Q_1 + Q_2 + Q_3$$

$Q_i$ is quantity of product sold in market $i$.

Find the quantity and price in each market $i$ that maximizes firm's profit.

**Step 1** State the objective Function: Profit function

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**Step 2** Find FONC and use FONC to find critical values

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$$MC = MR_1 = MR_2 = MR_3$$

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Conclusion: In the market with lower price elasticity of demand ( lower $|\varepsilon_{di}|$), the maximizing-profit price in that market will be...........

**Step 3** Check SOSC whether $Q_1^*, P_1^*, Q_2^*, P_2^*, Q_3^*, P_3^*$ indeed give the maximum profit

Hessian Matrix:

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}$$

$$H = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Check all leading principal minors

$|H_1| = |\pi_{11}|$

$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix}$

$|H_3| = \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{vmatrix}$

Example: Suppose a monopolist firm sells its product in three markets with corresponding market demand:

*Domestic market*: $P_1 = 63 - 4Q_1$
*Asia market:* $P_2 = 105 - 5Q_2$
*Europe market:* $P_3 = 75 - 6Q_3$

Total cost function of this firm is:

$$C = 20 + 15Q;$$
$$Q = Q_1 + Q_2 + Q_3$$

Find total quantity produced by the monopolist and price and quantity in each market that maximize profit.

**Step 1** State the objective Function: Profit function

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**<u>Step 2</u>** Find FONC and use FONC to find critical values

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**<u>Step 3</u>:** SOSC

$$H = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

Check all leading principal minors

$|H_1| =$

$|H_2| =$

$|H_3| =$

**Competitive Firm Input Choices: Cobb-Douglas Technology**

Let a firm in competitive market have Cobb-Douglas Production Function:

$$Q = L^{\alpha} K^{\beta}$$

Let wage be $w$, rent be $r$ baht per unit.

Characteristics of the production function:

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**<u>Step 1</u>** State the objective Function: Profit function

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**<u>Step 2</u>** Find FONC and use FONC to find critical values

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**The Economic interpretation**

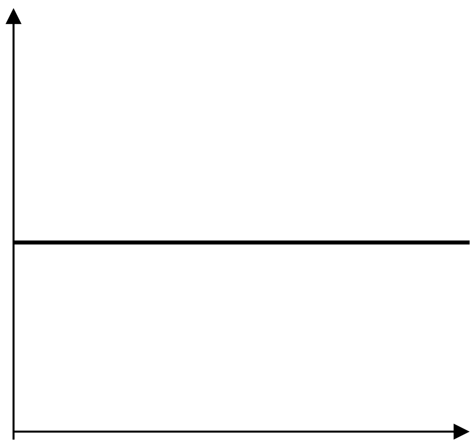
**Step 3:** SOSC

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

Check all leading principal minors

$|H_1| =$

$|H_2| =$



## Multiproduct Firm

### Multiproduct Firm in competitive market

Let a firm produce and sell two products. Each product is sold in competitive market, hence the firm is price taker. Total Revenue is:

$TR =$

$P_1$ is market price of good 1.
$P_2$ is market price of good 2.
$Q_1$ is quantity of good 1 sold in the market.
$Q_2$ is quantity of good 2 sold in the market.

Total cost of this firm is:

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: ……………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of $\pi$, $d^2\pi$, is .......................definite. Hence, the profit is………………..

**Multiproduct Firm in monopoly market**

Let market demand of good 1 is $Q_1 = 40 - 2P_1 + P_2$
Let market demand of good 2 is $Q_2 = 15 + P_1 - P_2$

From market demand, good 1 and good 2 are ....................................

The market demands are given as functions of $P_1, P_2$. Since we would like to choose $Q_1, Q_2$, we need to convert the above market demands to be in the form of $P_1 = f(Q_1, Q_2)$, and $P_2 = g(Q_1, Q_2)$.

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Then, we can use the inverse market demand functions to get total revenue function:

$TR =$…………………………………………………………………………………………

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Let the cost function of the monopolist be

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

Profit Function:

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of $\pi$, $d^2\pi$, is ......................definite. Hence, the profit is………………..

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**Multiplant Firm Problem**

Consider a firm with many plants. Each plant $i$ has total cost $TC_i$.

$$TC_i = C_i(q_i),\ i = 1,2,\dots,n$$

$q_i$ is the level of output produced by factory $i$.

Assume that this firm sells its product in one market. Firm's total revenue function is:

$TR =$………………………………………………………………………………

$Q =$…………………………………

If output market is competitive market,

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If output market is monopoly market,

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: …………………………………………………

*NOTE: Third-degree price discrimination vs. Multiplant firm*

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First Order Necessary Condition:

$\pi_i$ = ……………………………………………………………………for all $i = 1,2, \ldots, n$

SOSC:

$$H = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$\left|H_1\right| = R'' - C_1'' < 0$

$\left|H_2\right| =$

**Economic interpretation:**

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Example: Consider a monopolist firm with two plants:

Factory 1: $C_1(Q_1) = 10Q_1^2$

Factory 1: $C_2(Q_2) = 20Q_2^2$

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Firm's market demand is $P = 700 - 5Q$, $Q = Q_1 + Q_2$

Draw $MC_1, MC_2, MC_T, AR, MR$ and indicate the maximizing level of output from each plant and the profit-maximizing price.

Find $Q_1$, $Q_2, Q_{Total}, P$ that maximize profit of this firm.

Profit Function:

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of $\pi$, $d^2\pi$, is .......................definite. Hence, the profit is………………..

…………………………………………………………………………………………………………

*Summary:*

*3rd price discrimination:*

……………………………………………………………………………………………………

*Choosing levels of factor of production:*

……………………………………………………………………………………………………

*Multiproduct:*

……………………………………………………………………………………………………

*Multiplant:*

……………………………………………………………………………………………………