



B.E. International Program

Faculty of Economics, Thammasat University



EE 320 Introductory Mathematical Economics

Semester 2/2013

Practice Problem 9

(Optimization Problems with 1-Equality Constraint)^{1,2}

Suggested Answers

1. Suppose that Mr. Pisut's utility depends on two commodities: x_1 and x_2 . His utility function is given by: $U = x_1x_2 + 2x_1 + x_2$.

Suppose that Pisut's income is \$76, and the per-unit prices for the commodities x_1 and x_2 are \$2 and \$3, respectively.

- (a) Determine the values x_1^* and x_2^* that maximize Pai's utility, given that Pai spent all of his income on these two commodities.

Ans.

$$\begin{aligned} \text{Max}_{x_1, x_2} \quad & U(x_1, x_2) = x_1x_2 + 2x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 = 76 \end{aligned}$$

$$\begin{aligned} \mathcal{L}(x_1, x_2, \lambda) &= x_1x_2 + 2x_1 + x_2 + \lambda[76 - 2x_1 - 3x_2] \\ \text{FOC: } \mathcal{L}_1 = \frac{\partial \mathcal{L}}{\partial x_1} &= x_2^* + 2 - 2\lambda = 0 \quad \text{--- ①} \end{aligned}$$

$$\mathcal{L}_2 = \frac{\partial \mathcal{L}}{\partial x_2} = x_1^* + 1 - 3\lambda = 0 \quad \text{--- ②}$$

$$\mathcal{L}_\lambda = \frac{\partial \mathcal{L}}{\partial \lambda} = 76 - 2x_1^* - 3x_2^* = 0 \quad \text{--- ③}$$

$$\Rightarrow x_1^* = 20 \text{ and } x_2^* = 20$$

¹ Questions 4-11 are from Sydsaeter and Hammond, 2008

² *, ** indicate levels of difficulty.

(b) Show the second-order sufficient condition for the constrained utility maximization.

Ans.

$$\text{S.O.S.C: } L_{11} = 0 ; L_{21} = L_{12} = 1 ; L_{22} = 0 \\ g_1 = 2 ; g_2 = 3$$

$$|H| = \begin{vmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} = (-2) \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} + (3) \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = (-2)(-3) + (3)(2-0) \\ = 6 + 6 = 12 > 0.$$

Thus, $U^* = 720$ is a maximum utility given the budget of \$76.

2. Given the production function

$$Q = f(K, L) = K^{0.5} L^{0.5}$$

Suppose that the per unit input prices for K and L are \$20 and \$5, respectively.

(a) If the producer needs to maintain the output level at $\bar{Q} = 20$ units, what are the values K^* and L^* that minimizes the total cost, and what is the corresponding minimum cost? Verify that the second-order sufficient condition is met.

Ans.

$$\text{Min}_{K, L} TC = rK + wL = 20K + 5L \\ \text{s.t. } K^{0.5} L^{0.5} = 20$$

$$\Rightarrow L(K, L, \lambda) = 20K + 5L + \lambda [20 - K^{0.5} L^{0.5}]$$

F.O.N.C.

$$L_K = 20 - (\lambda)(0.5 K^{-0.5} L^{0.5}) = 0 \quad \text{--- (1)}$$

$$L_L = 5 - (\lambda)(0.5 K^{0.5} L^{-0.5}) = 0 \quad \text{--- (2)}$$

$$L_\lambda = 20 - K^{0.5} L^{0.5} = 0 \quad \text{--- (3)}$$

$$\Rightarrow K^* = 10; L^* = 40; TC^* = 400; \text{ and } \lambda^* = 20$$

S.O.S.C.

$$|H_2| = \begin{vmatrix} 0 & 1 & 1/4 \\ 1 & 1 & -1/4 \\ 1/4 & -1/4 & 1/16 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -1/4 \\ 1/4 & 1/16 \end{vmatrix} + (1/4) \begin{vmatrix} 1 & 1 \\ 1/4 & -1/4 \end{vmatrix} \\ = -1 - \frac{1}{8} = -\frac{9}{8} < 0$$

$\therefore$ SOSC for a minimum TC is satisfied.

- (b) Suppose now that the output level is not fixed, but the producer has a budget constraint at \$400. Assume that this producer spends his entire budget on the production. Verify that the values K^* and L^* that maximizes the total output are the same as the values found in part (a).

Ans.

$$\begin{aligned} \text{Max}_{K, L} \quad Q &= K^{0.5} L^{0.5} \\ \text{s.t.} \quad 20K + 5L &= 400 \\ \mathcal{L}(K, L, \lambda) &= K^{0.5} L^{0.5} + \lambda [400 - 20K - 5L] \\ \text{FONC:} \\ \mathcal{L}_K &= 0.5 K^{-0.5} L^{0.5} - 20\lambda = 0 \quad \text{--- (1)} \\ \mathcal{L}_L &= 0.5 K^{0.5} L^{-0.5} - 5\lambda = 0 \quad \text{--- (2)} \\ \mathcal{L}_\lambda &= 400 - 20K - 5L = 0 \quad \text{--- (3)} \end{aligned}$$

$\Rightarrow K^* = 10$ and $L^* = 40$, which are the same as the values found from the cost minimization problem.

- *3. Suppose that a monopolistic firm sells a single product in three separate markets (say, three different countries), and therefore faces three different demand functions:

$$P_1 = 32 - 1.5Q_1$$

$$P_2 = 68 - 3Q_2$$

$$P_3 = 40 - 2Q_3$$

The total cost function is given by: $TC = 60 + 20Q$, where $Q = Q_1 + Q_2 + Q_3$.

- a) Suppose that there is a shortage of a key raw material so that the total production must be restricted to $\bar{Q} = 12$ units. Find the profit-maximizing output levels of Q_1 , Q_2 , and Q_3 , and determine the maximum profit.

Ans. $(Q_1^*, Q_2^*, Q_3^*) = (\frac{16}{9}, \frac{62}{9}, \frac{10}{3})$ and $\pi^* = 189.33$

- b) Verify that the second-order sufficient condition is met.

$$\text{Ans. } |\bar{H}| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & -6 & 0 \\ 1 & 0 & 0 & -4 \end{vmatrix}; |\bar{H}_2| < 0; |\bar{H}_3| > 0$$

4. a) A firm produces two different kinds A and B of a commodity. The daily cost of producing x units of A and y units of B is

$$C(x, y) = 2x^2 - 4xy + 4y^2 - 40x - 20y + 514$$

Suppose that the firm sells all its output at a price per unit of \$24 for A and \$12 for B. Find the daily production levels of x and y that maximize profit.

Ans. $(x^*, y^*) = (40, 24)$

b) The firm is required to produce exactly 54 units per day of the two kinds combined. What now is the optimal production plan?

Ans. $(x^{**}, y^{**}) = (34, 20)$.

[Hint: Since $x + y = 54$, substitute $y = 54 - x$ in the profit function and then solve for the optimal values of x and y , respectively.]

5. Suppose that the utility maximization problem is

$$\max U = xyz \text{ subject to } x + 3y + 4z = 108.$$

a) Solve the utility maximization problem by making U a function of y and z by eliminating the variable x .

Ans. Rewrite the problem as: $\max U = (108 - 4y - 4z)yz$. Then, take find $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$, and it follows that $(x^*, y^*, z^*) = (36, 12, 9)$.

b) Solve the above problem by using the Lagrange multiplier method.

Ans. You should get the same answer as in part a).

6. The profit obtained by a firm from producing and selling x and y units of two brands of a commodity is given by

$$P(x, y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$$

a) Find the production levels that maximize profits.

Ans. $(x^*, y^*) = (230, 5)$

b) A key raw material is rationed so that total production must be restricted to 200 units. Find the production levels that now maximize profits.

Ans. $(x^{**}, y^{**}) = (195, 5)$.

7. Use the Lagrange's method to solve the problem

$$\min_{Q_1, Q_2} -40Q_1 + Q_1^2 - 2Q_1Q_2 - 20Q_2 + Q_2^2 \text{ Subject to } Q_1 + Q_2 = 15.$$

Use the bordered Hessian to verify that the second-order sufficient conditions are satisfied.

Ans. $\mathcal{L} = -40Q_1 + Q_1^2 - 2Q_1Q_2 - 20Q_2 + Q_2^2 + \lambda[15 - Q_1 - Q_2]$.

$\rightarrow (Q_1^*, Q_2^*) = (10, 5)$.

$|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 2 \end{vmatrix} = -8 < 0$. Thus, SOSC for a minimum are satisfied.

8. A person has utility function

$$u(x, y) = 100xy + x + 2y.$$

Suppose that the price per unit of x is \$2, and that the price per unit of y is \$4. The person receives \$1000 that all has to be spent on the two commodities x and y . Solve the utility maximization problem, and verify that the second-order sufficient conditions are satisfied.

Ans. $(x^*, y^*) = (250, 125)$

$|\bar{H}| = \begin{vmatrix} 0 & 2 & 4 \\ 2 & 0 & 100 \\ 4 & 100 & 0 \end{vmatrix} = 1600 > 0$. Thus, SOSC for a maximum are satisfied.

9. A firm produces and sells two commodities. By selling x tons of the first commodity the firm gets a price per ton given by $p = 96 - 4x$. By selling y tons of the other commodity the price per ton is given by $q = 84 - 2y$. The cost of producing and selling x tons of the first commodity and y tons of the second is given by $C(x, y) = 2x^2 + 2xy + y^2$.

a) Show that the firm's profit function is $P(x, y) = -6x^2 - 3y^2 - 2xy + 96x + 84y$.

Ans. $P(x, y) = (96 - 4x)x + (84 - 2y)y - (2x^2 + 2xy + y^2)$
 $= -6x^2 - 3y^2 - 2xy + 96x + 84y$

b) Compute the first-order partial derivatives of P , and find its only stationary point.

Ans. $P_x = -12x - 2y + 96$; $P_y = -6y - 2x + 84$

$(x^*, y^*) = (6, 12)$ and $P(x^*, y^*) = 792$.

c) Suppose the production causes pollution, and that the authorities for this reason require the firm to produce only 11 tons in total. Solve the firm's maximization problem in this case. Verify that the production restrictions do reduce the maximum possible value of $P(x, y)$.

Ans. $(x^{**}, y^{**}) = (4, 7)$; $P(x^{**}, y^{**}) = 673$.

10. a) Solve the utility maximization problem

$$\max_{x, y} U(x, y) = \sqrt{x} + y \quad \text{subject to} \quad x + 4y = 100$$

using the Lagrange method, i.e. find the quantities demanded of the two goods.

Ans. $(x^*, y^*) = (4, 24) \Rightarrow U(x^*, y^*) = 26$ [Note: $\lambda = 0.25$]

b) Suppose income increases from 100 to 101. What is the exact increase in the optimal value of $U(x, y)$? Compare with the value found in (a) for the Lagrange multiplier.

Ans. $(x^{**}, y^{**}) = (4, 24.25) \Rightarrow U(x^{**}, y^{**}) = 26.25$.

Thus, $U(x^{**}, y^{**}) - U(x^*, y^*) = 0.25 = \lambda$.

c) Suppose we change the budget constraint to $px + qy = m$, but keep the same utility function. Derive the quantities demanded of the two goods if $m > q^2/4p$.

Ans. $x^* = \frac{q^2}{4p^2}$; $y^* = \frac{m}{q} - \frac{q}{4p}$. Note: $y^* > 0 \Leftrightarrow m > q^2/4p$.

**11. a) Consider the consumer's demand problem

$$\max_{x,y} [U(x,y) = \alpha \ln(x-a) + \beta \ln(y-b)] \quad \text{subject to} \quad px + qy = m \quad (*)$$

Where $\alpha, \beta, a, b, p, q$, and m are positive constants with $\alpha + \beta = 1$, and moreover, with $m > ap + bq$. Show that if x^*, y^* solve the problem (*), then expenditure on the two goods is given by the two linear functions

$$px^* = \alpha m + pa - \alpha(pa + qb); \quad qy^* = \beta m + qb - \beta(pa + qb) \quad (**)$$

of the variables (m, p, q) .

Ans. From FONC:

$$x^* = a + \frac{\alpha}{\lambda p}; \quad y^* = b + \frac{\beta}{\lambda q}; \quad \text{and} \quad \frac{1}{\lambda} = m - (pa + qb).$$

Thus, it follows that

$$px^* = \alpha m + pa - \alpha(pa + qb); \quad qy^* = \beta m + qb - \beta(pa + qb).$$

b) Let $U^*(p, q, m) = U(x^*, y^*)$ denote the indirect utility function. Show that $\partial U^*/\partial m > 0$ and verify the so-called Roy's identities, $\frac{\partial U^*}{\partial p} = -\frac{\partial U^*}{\partial m} x^*$ and $\frac{\partial U^*}{\partial q} = -\frac{\partial U^*}{\partial m} y^*$.

Ans.
$$\frac{\partial U^*}{\partial m} = \frac{\alpha}{m-(pa+qb)} + \frac{\beta}{m-(pa+qb)} = \frac{1}{m-(pa+qb)} = \lambda > 0;$$

$$\frac{\partial U^*}{\partial p} = \frac{-\alpha a}{m-(pa+qb)} - \frac{\alpha}{p} + \frac{-\beta a}{m-(pa+qb)} = \frac{-a}{m-(pa+qb)} - \frac{\alpha}{p} = -a\lambda - \frac{\alpha}{p}.$$

Also, $-\frac{\partial U^*}{\partial m} x^* = -\lambda \left(a + \frac{\alpha}{\lambda p} \right) = -a\lambda - \frac{\alpha}{p}$. Thus, $\frac{\partial U^*}{\partial p} = -\frac{\partial U^*}{\partial m} x^*$.

$\frac{\partial U^*}{\partial q} = -\frac{\partial U^*}{\partial m} y^*$ can be shown in the same manner.