Chapter 6 Relations and Functions

6.1 Order pairs and Cartesian product of A and B

Definition 6.1.1: ***Cartesian product of A and B***, denoted by $A \times B$, is the set of all order pairs which is written as a coordinate (x, y) where the first element of each order pair is an element of *A* and the second element of each order pair is an element of *B*.

Theorem 6.1.1: Let (x, y) and (a, b) be any order pairs. If $(x, y) = (a, b)$ then $x = a$ and $y = b$.

Example 6.1.1: Let $A = \{1, 2\}$ and $B = \{-1, 0, 9\}$. Find $A \times B$, $B \times A$, $A \times A$, and $B \times B$.

6.2 Relations and Functions

Definition 6.2.1: A rule that assigns elements of B to elements of A is called a relation from A to B. The set of elements of A to which the assignments are made is called the domain of the relation. The set of elements of B used in the assignments is the range of relation.

Definition 6.2.2: A relation from set A to set B is called a mapping or a function if the relation assigns to each element of A exactly one element of B.

Theorem 6.2.1: Vertical-Line Test for a Function

An equation defines a function if each vertical line in the coordinate system passes through at most one point on the graph of the equation. If any vertical line passes through two or more points on the graph of an equation, then the equation does not define a function.

Definition 6.2.3: f is said to be a function from A **into** B if f is a function that has A as a domain and range of f is a subset of B. A function A into B is denoted by $f : A \to B$ or $f : A \xrightarrow{\text{into}} B$.

Definition 6.2.4: f is said to be a function from A **onto** B if f is a function from A into B and range of f is B ($R_f = B$). A function A onto B is denoted by $f : A \xrightarrow{\text{onto}} B$.

Example 6.2.1: Let $A = \{1, 2\}$ and $B = \{-1, 0, 9\}$. Are the following relations a function?

1) $r = \{(x, y) \in A \times B \mid x + y = x\}$
2) $t = \{(x, y) \in B \times A \mid 0 < x + y \le 10\}$
3) $s = \{(x, y) \in B \times B \mid xy \ge 0\}$
4) $u = \{(x, y) \in A \times A \mid 3 \le 2x + y \le 4\}$

Example 6.2.2: Let $A=\{0,1,2\}$ and $B=\{0,1,4\}$. Are the following relations a function?

1) $f=\{(x,y)\in A\times B \mid y=x^2\}$

2) $g=\{(x,y)\in A\times B \mid 2x+y=2\vee 2x+y=4\}$

Example 6.2.3: Check whether the following relation is a function. If it is a function then explain that it is a function from where onto where.

1) $f_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y + x = 3\}$

2) $f_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = \sqrt{x-1}\}$

3) $f_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x + y^2 = 3\}$

Definition 6.2.5: f is said to be one-to-one function from A into B, denote by $f : A \xrightarrow{1-1} B$, if f is a function from A into B and $\left[(x_1, y) \in f \wedge (x_2, y) \in f\right] \rightarrow (x_1 = x_2)$.

Definition 6.2.6: f is said to be one-to-one function from A onto B, denote by $f : A \xrightarrow[onto]{1-1} B$, if f is a function from A onto B and f is one-to-one.

Note: A function is a ***one-to-one function*** if and only if each second element corresponds to one and only one first element. (each x and y value is used only once)

Example 6.2.4: Recall Example 6.2.3. Consider the following relations and check if they are one-to-one functions.

1) $f_1 = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \middle| y + x = 3\right\}$

2) $f_2 = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \middle| y = \sqrt{x - 1}\right\}$

Definition 6.2.7: The *inverse* of a function is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original function. For function f , the inverse of the function is denoted by f^{-1}.

Note: If the original function is a one-to-one function, the inverse will be a function.

Example 6.2.5: Let $A=\{1,2,3,5\}$, $B=\{-2,-1,0,4\}$ and let f and g be functions from A to B where $f=\{(1,0),(2,-1),(3,-2),(5,0)\}$ and $g=\{(1,-2),(2,-1),(3,0),(5,4)\}$. Find f^{-1} and g^{-1}

Example 6.2.6: Let $f = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} \,\middle|\, y = 2x - 3 \right\}$ be a function. Find the inverse of a function f and check if it is a function.

Example 6.2.7: Let f be a function defined by $f(x)=1+\sqrt{x-2}$. Find f^{-1} and check if it is a function.

Example 6.2.8: Let g be a function defined by $g\left(x\right)=\begin{cases}1-x & ,x<0\\ x^2+1 & ,x\ge 0\end{cases}$.

Is g^{-1} a function? Find the domain and range of g^{-1}.