4 Testing Hypotheses about a Single Linear Combination of the Parameter

Consider

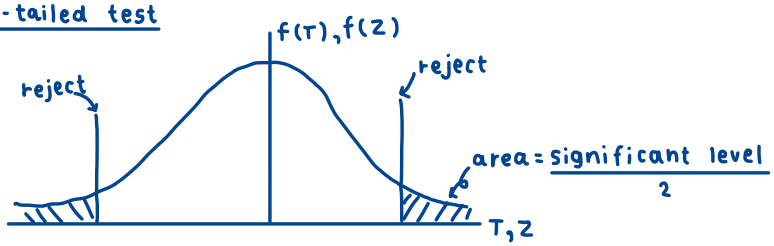
$$\log(wage) = \beta_0 + \beta_1 jc + \beta_2 univ + \beta_3 \exp er + u$$

where jc = number of years attending a two-year college
$univ$ = number of years at a four-year college
$\exp er$ = months in the workforce.

We want to test whether $\beta_1 = \beta_2$. → If the returns from 1 more year of education at a junior college is the same as that of the university.

$H_0: \beta_1 = \beta_2 \rightarrow H_0: \beta_1 - \beta_2 = 0$
against
$H_a: \beta_1 \neq \beta_2 \rightarrow H_a: \beta_1 - \beta_2 \neq 0$

2-tailed test



$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{s.e.(\hat{\beta}_1 - \hat{\beta}_2)}$$ ← we compute this t-statistic and compare with the critical value.

where $s.e.(\hat{\beta}_1 - \hat{\beta}_2) = \sqrt{\widehat{var}(\hat{\beta}_1 - \hat{\beta}_2)}$

$$= \sqrt{\widehat{var}(\hat{\beta}_1) + \widehat{var}(\hat{\beta}_2) - 2\widehat{cov}(\hat{\beta}_1, \hat{\beta}_2)}$$

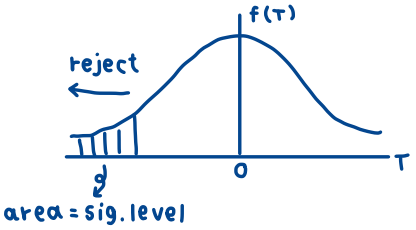
not very significant forward to calculate
→ we use a variable transformation trick (see notes)

another possible hypothesis test (one-tailed alternative)

$H_0: \beta_1 = \beta_2 \rightarrow H_0: \beta_1 - \beta_2 = 0$

$H_a: \beta_1 < \beta_2 \rightarrow H_a: \beta_1 - \beta_2 < 0$

•It is assume that β_1 would not be more than β_2 (return to a 2-year college would never be more than returns to university education)



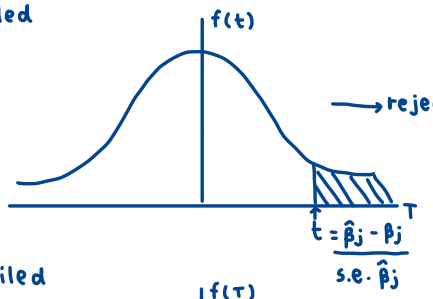
$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{s.e.(\hat{\beta}_1 - \hat{\beta}_2)}$$

*Then go to extra notes

5 Computing p-Values for t-Tests

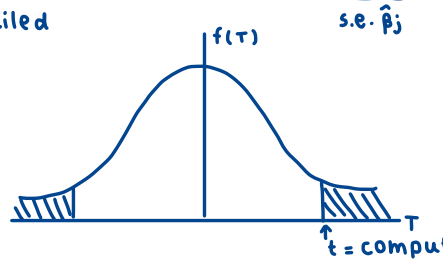
- What is the significance level given the computed t-statistics?

1-tailed



This shaded area in the rejection region is the p-value.

2-tailed



total area from 2 sides.

$$t = \frac{\hat{\beta}_j - \beta_j}{s.e.\hat{\beta}_j}$$

- p-value : $P(|T| > |t|)$

T = t-distributed random variable with d.f. = n-k-1

t = computed t-statistic

↝ p-value = probability that a random T value will be greater (in the | | term) than our t in the H_0 test.

In-class exercise see notes!!

consider the multiple regression model, assume MLR 1-6 are satisfied.

$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$

you would like to test the $H_0: \beta_1 - 3\beta_2 = 1$

H_a: Otherwise is true

1st) write the t-statistic for testing H_0

$$t = \frac{(\hat{\beta}_1 - 3\hat{\beta}_2) - 1}{s.e\ (\hat{\beta}_1 - 3\hat{\beta}_2)}$$

2nd) define $\theta_1 = \hat{\beta}_1 - 3\hat{\beta}_2 \rightarrow H_0: \theta_1 = 1$, $H_a: \theta \neq 1$

$$t = \frac{\hat{\theta}_1 - 1}{s.e.(\hat{\theta}_1)}$$ → we need our regression to have θ_1 in it. so, stata or OLS estimation will automatically give $\hat{\theta}_1$ & $s.e.\hat{\theta}_1$

Now, $\hat{\beta}_1 = \hat{\theta}_1 + 3\hat{\beta}_2$

$\beta_1 = \theta_1 + 3\beta_2$

sub in the main regression and get

$y = \beta_0 + (\theta_1 + 3\beta_2)x_1 + \beta_2 x_2 + \beta_3 x_3 + u$

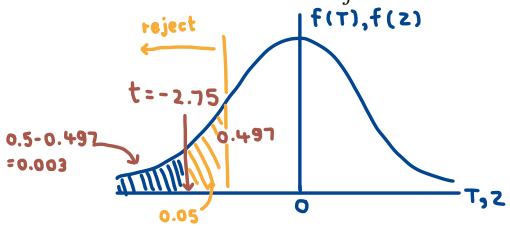
$= \beta_0 + \theta_1 x_1 + 3\beta_2 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$

$= \beta_0 + \theta_1 x_1 + \beta_2 (x_2 + 3x_1) + \beta_3 x_3 + u$

* Now, the explanatory variables are going to be $x_1, x_2 + 3x_1$, and x_3

- we can calculate $t = \frac{\hat{\theta}_1 - 1}{s.e.\hat{\theta}_1}$

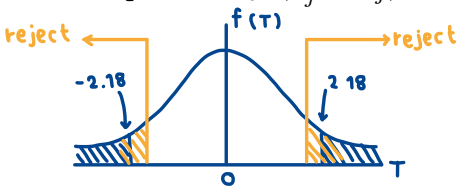
Example 1: $H_0 : \beta_j \geq 0, \quad H_a : \beta_j < 0, \quad$ d.f.$= 140$. → z-table
→ p-value = what should be the significant level, given the critical value of -2.75?? → find the shaded area



suppose the calculated $t_{\hat{\beta}_j} = -2.75$ → $t_{\hat{\beta}_j} = \frac{(\hat{\beta}_j - \beta_j)}{s.e.(\hat{\beta}_j)}$

- From the z-table, the value -2.75 corresponds to area = _ _ _0.003_ _ _ _ _ _ _ _.
- Thus, p-value = _ _ 0.003 _ _ _ _ _ _ _.
- Would we reject H_0 if we use the significance level = 5%? YES
  *RULE: we reject H_0 if p-value < sig. level

Example 2: $H_0 : \beta_j = a_j, \quad H_a : \beta_j \neq a_j, \quad$ d.f.$= 18$. ↖ use t-table

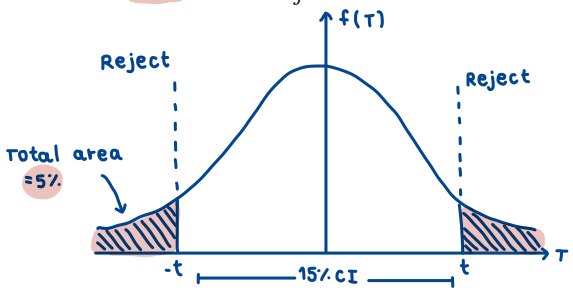


suppose the calculated $t_{\hat{\beta}_j} = -2.18$

- From the t-table, the value -2.18 corresponds to area = _0.02+0.05 _ _ _ _ _ _ _ _.
- Thus, p-value = is _between_ 0.02 - 0.05 _.
- Would we reject H_0 if we use the significance level = 5%?
  yes, reject H_0 because the area is less than 0.05 OR p-value < 0.05

6 Confidence Intervals (CI)

- Confidence Intervales for the POPULATION PARAMETER (β_j)
  → The range of values that would capture the true β_j at a 15% chance
- A 95% CI of β_j is given by



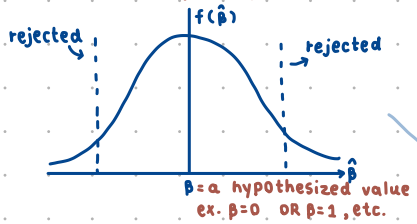
CI → $\hat{\beta}_j \pm c \times s.e.(\hat{\beta}_j)$
c is the 97.5 percentile in the t-distribution with n-k-1 d.f.

Inference → Hypothesis testing about "β" the true parameter.

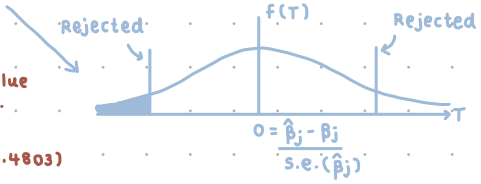
$wage = \beta_0 + \beta_1 educ + \beta_2 experience + \dots + u$

we want to test hypothesis about the true impact (β) of each x variables (edu, experience) on the dependent variable (y)

But! we don't know what the true β are. So, we use $\hat{\beta}$ (estimator) and s.e.($\hat{\beta}$) to test the hypothesis.



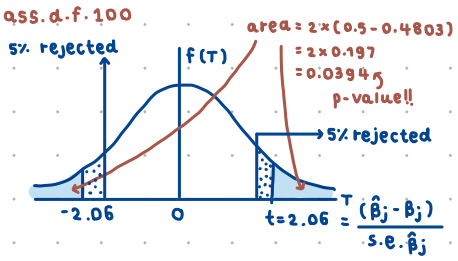
1) Test if β = some number e.g. $\beta_j = 0 \rightarrow x_j$ has no impact on y.
$\beta_j = 1 \rightarrow$ 1 unit↑ in x_j correspond to 1 unit↑ in y.



→T-test ✱ How??

$\frac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)} \sim t_{d.f.}$

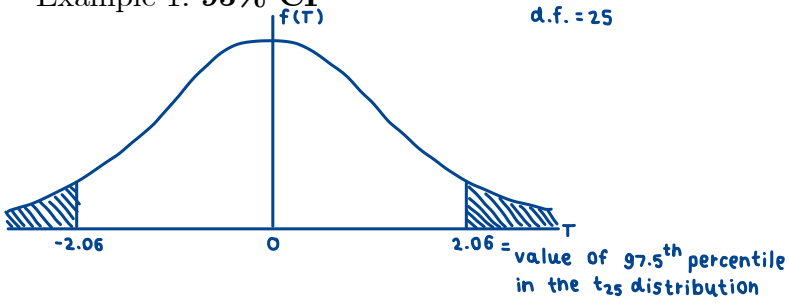
significant level = total area in the rejection region



- Suppose, we calculate a t-statistic = $\frac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)} = 5.78$
- suppose, we are testing $H_0: \beta_j = 0$, $H_a: \beta_j \neq 0$ → 2 tailed test
- p-value = total shaded area

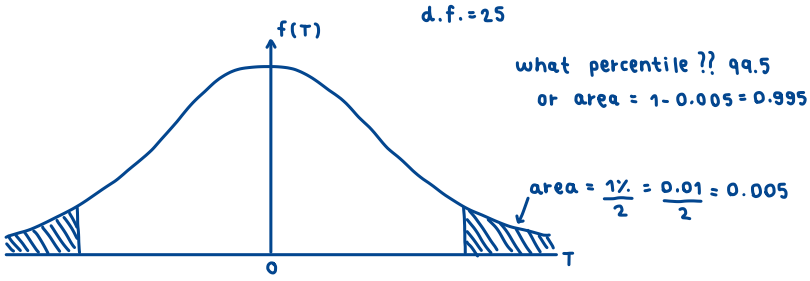
p-value = significant level which we will reject the H_0 OR Prob that we will reject H_0.
✱ If p-value < significance level → reject H_0

Example 1: **95% CI**



The 95% CI for $\hat{\beta}_j = [\hat{\beta}_j - 2.06 \cdot s.e(\hat{\beta}_j), \hat{\beta}_j + 2.06 = s.e.(\hat{\beta}_j)]$

Example 2: **99% CI**



The 99% CI for $\hat{\beta}_j = [\hat{\beta}_j - 0.787 \cdot s.e\hat{\beta}_j, \hat{\beta}_j + 0.787 \cdot s.e(\hat{\beta}_j)]$

F-test motivation

- we want to test the significance of a group of hypothesis (multiple hypothesis)

$Grade_{325} = \beta_0 + \beta_1 \#times\_front + \beta_2 \#times\_back + \beta_3 hr\_study + \beta_4 past\_GPA + \beta_5 gender + u$

H_0: seat position doesn't have impact on GPA

$\beta_1 = 0$ and $\beta_2 = 0 \rightarrow \beta_1 = \beta_2 = 0$

H_a: seat position matters

$\beta_1 \neq 0$ and $\beta_2 \neq 0$

or $\beta_1 \neq 0$ and $\beta_2 = 0$

or $\beta_1 = 0$ and $\beta_2 \neq 0$

} at least one of the $\beta_1, \beta_2 \neq 0$

7 Testing Multiple Linear Restrictions: The F-test

Suppose the model is specified by

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

H_0 : $\beta_2 = 0$ and $\beta_3 = 0$ → want to test if x_2 and x_3 BOTH have no impact on y.

H_a, H_1 : H_0 is not true

We can use the F-test to test this type of "multiple hypotheses".

1. Our full model is called the "unrestricted" model (ur). Suppose it can be expressed as: ← Big model

→ $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$ is true → Reject H_0

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_k x_k + u$$

2. The model which takes out x (which we think its associated $\beta = 0$) is called the restricted model (r). ← small model

→ $y = \beta_0 + \beta_1 x_1 + u$ is true → do not reject H_0

suppose there are "q" number of β that we would like to perform a joint-test of $=0$

e.g. in this model $q = 2$

$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{k-q} x_{k-q} + u$

$H_0 : \beta_{k-q+1} = \beta_{k-q+2} = \beta_k = 0$

(the last q $\beta_s = 0$)

H_a: H_0 is not true

$$y = \underbrace{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{k-q} x_{k-q}}_{(r)} + \underbrace{\beta_{k-q-1} x_{k-q-1} + \beta_{k-q-2} x_{k-q-2} + \ldots + \beta_k x_k + u}_{ur}$$

$$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}$$

This is always (+) b/c $SSR_{ur} < SSR_r$ every time you add 1 more x, the model will be better explained

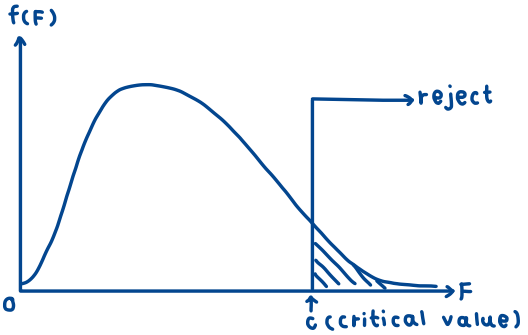
$(n-k-1)$ d.f. of the "ur" model.

- So, if every time you add 1 more x variable, the SSR ↓ and R^2 ↑, why don't we just keep the additional x in the model??

→ Because every time we add 1 more x, $var(\hat{\beta}_s)$ will increase, making the prediction of β less precise. So, we only keep the addition x_s if it/they can improve the model enough.

can ↓ SSR (↑ R^2) enough.

can significantly ↓ SSR and ↑ R^2.



$H_0 : \beta_2 = \beta_3 = \ldots = 0$

H_a: H_0 ! not true

$F \sim F_{q, n-k-1}$ ← d.f. of the ur model

# of joint Hypothesis being tested

3. Some useful facts

1) $R^2_{ur} > R^2_r$ because any additional x would increase R^2 (improve fit)
>> $SSR_{ur} < SSR_r$

2) By including more x, the model is certainly better explained.
However, we would like to reject H_0 if the inclusion of extra variables does not improve the model enough.

4. Other ways to calculate the F-statistics:

From $R^2 = 1 - \frac{SSR}{SST}$ (SSR → RSS, SST → TSS)

we have $F \equiv \frac{(R^2_{ur} - R^2_r)/q}{(1 - R^2_{ur})/(n-k-1)}$

q: # of β that are set to "0"

n: # of obs; k: # of slope β; (k+1) → Intercept

>> If we want to test the overall significant of the model
$H_0: \beta_1 = \beta_2 = \beta_3 = \ldots = \beta_k = 0$, H_a = otherwise

$F \equiv \frac{R^2/k}{(1-R^2)/(n-k-1)}$

R^2 of the model ≈ u_R
the "r" model has no x at all.

Example: Suppose we are interested in understanding the determinant of a baseball player's salary.

y	*salary*	= season salary
r, ur	*years*	= years in major leagues
ur	*gamesyr*	= games per year in the league
ur	*bavg*	= career batting average
ur	*hrunsyr*	= homeruns per year
ur	*rbisyr*	= runs batted in per year

If we want to test whether performance has any impact on salary
$H_0: \beta_{bavg} = \beta_{hrunsyr} = \beta_{rbisyr} = 0$
H_a: otherwise is true

- the unrestricted model (ur) is defined by

ur model

```
. regress log_salary years gamesyr bavg hrunsyr rbisyr
```

Source	SS	df	MS
Model	308.989208	5	61.7978416
Residual	183.186327	347	.527914487
Total	492.175535	352	1.39822595

Number of obs	= 353
F(5, 347)	= 117.06
Prob > F	= 0.0000
R-squared	= 0.6278
Adj R-squared	= 0.6224
Root MSE	= .72658

log_salary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
years 1	.0688626	.0121145	5.68	0.000	.0450355	.0926898
gamesyr 2	.0125521	.0026468	4.74	0.000	.0073464	.0177578
bavg 3	.0009786	.0011035	0.89	0.376	-.0011918	.003149
hrunsyr 4	.0144295	.016057	0.90	0.369	-.0171518	.0460107
rbisyr 5	.0107657	.007175	1.50	0.134	-.0033462	.0248776
_cons intc.	11.19242	.2888229	38.75	0.000	10.62435	11.76048

$q = 3$

when considering each of the performance x one-by-one, none of them has a significance impact at 5%

- the restricted model (r) is defined by

```
. regress log_salary years gamesyr
```

Source	SS	df	MS
SSE Model	293.864058	2	146.932029
SSR Residual	198.311477	350	.566604221
SST Total	492.175535	352	1.39822595

Number of obs	= 353
F(2, 350)	= 259.32
Prob > F	= 0.0000
R-squared	= 0.5971
Adj R-squared	= 0.5948
Root MSE	= .75273

log_salary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
years	.071318	.012505	5.70	0.000	.0467236	.0959124
gamesyr	.0201745	.0013429	15.02	0.000	.0175334	.0228156
_cons	11.2238	.108312	103.62	0.000	11.01078	11.43683

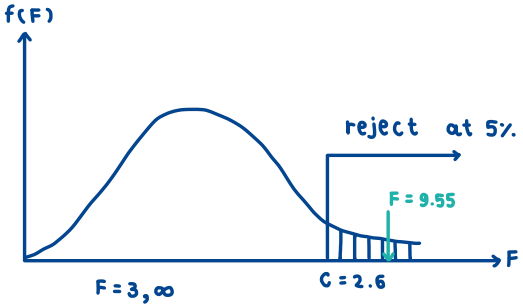
•But when performing an F-test, Performance have joint impact.

Now, our H_0 and H_a becomes

$$F \equiv \frac{\frac{SSR_r - SSR_{ur}}{q}}{\frac{SSR_{ur}}{n-k-1}} \equiv \frac{\frac{198.311 - 183.186}{3}}{\frac{183.186}{353-5-1}} \approx 9.55$$

Hw.

$$F \equiv \frac{\frac{R^2}{q}}{\frac{1-R^2}{n-k-1}} \equiv ?$$



since F = 9.55 > 2.6, we reject H_0 at 5% sig level and conclude that performances have joint effects on salary.

8 How the Hypothesis Testing is done in Practice

1. Check the values of $t-statistic$ reported by the statistical software (i.e. STATA, SPSS, SAS)

   $\Rightarrow$ These $t-statistics$ are to test $H_0 : \beta_i = 0$

   $\Rightarrow$ If the d.f. > 30, then when $t > 1.96$, we can reject H_0 with 5% sig. level (z-table)

   $\Rightarrow$ **When $t > 1.96$**, we can say that β_i is **statistically significant** at 5% level. (value of $\beta_i \neq 0$)

   $\Rightarrow$ **When $t < 1.96$** we can say that β_i is **not statistically significant** at 5% level.

   $\Rightarrow$ If $t < 1.96$ we can drop x_i from the model

   $\Rightarrow$ After we drop x_i, we estimate the new regression function and obtain a new set of $\hat{\beta}$.

2. We can also perform other hypothesis testings of interest.

   e.g. $H_0 : \beta_i = \beta_j$

   or $H_0 : \beta_i = 5$ etc.

   or perform an F-test for testing multiple linear restrictions

3. Usually, in economics, the estimation results are reported using this form

Dependent Variable: log(*salary*)			
Independent Variables	**(1)**	**(2)**	**(3)**
log(*sales*)	.224 (.027)	.158 (.040)	.188 (.040)
log(*mktval*)	——	.112 (.050)	.100 (.049)
profmarg	——	−.0023 (.0022)	−.0022 (.0021)
ceoten	——	——	.0171 (.0055)
comten	——	——	−.0092 (.0033)
intercept	4.94 (0.20)	4.62 0.25)	4.57 (0.25)
Observations *R*-squared	177 .281	177 .304	177 .353

sales → log(*sales*)

other company performance → log(*mktval*), *profmarg*

CEO characteristic → *ceoten*, *comten*

like a simple regression w/ 1x

This is page
Printer: Opa

Multiple Regression Analysis : Further Issues

1 Data scaling on OLS statistics

When we change the unit of measurment of a variable, the value of estimators would chance accordingly. For example

$$\widehat{bweght}_g = \widehat{\beta}_0 + \widehat{\beta}_1 cigs + \widehat{\beta}_2 fa\min c,$$

where
$bwght$ = child birth weight, in grams.
$cigs$ = number of cigarettes smoked by the mother while pregnant, per day.
$fa\min c$ = annual family income, in thousands of dollars.

- what if we use bweght in kilograms??

1 kg = 1000g

$$\widehat{bweght}_{kg} = \frac{\widehat{bweght}_g}{1000} = \frac{\hat{\beta}_0}{1000} + \frac{\hat{\beta}_1}{1000} cigs + \frac{\hat{\beta}_2}{1000} famin_c$$

$$= \hat{\alpha}_0 + \hat{\alpha}_1 cig_s + \hat{\alpha}_2 fa\,min_c$$

$$\Rightarrow \hat{\alpha}_0 = \frac{\hat{\beta}_0}{1000}, \quad \hat{\alpha}_1 = \frac{\hat{\beta}_1}{1000}, \quad \hat{\alpha}_2 = \frac{\hat{\beta}_2}{1000}$$

- what if we use faminc in USD (instead of 1000 USD)

$$bweght_g = \hat{\beta}_0 + \hat{\beta}_1 cigs + \hat{\beta}_2 faminc_{USD}$$

The value of this variable is going to be 1000 times larger than faminc.

$$\Rightarrow \hat{\theta}_2 = \frac{\hat{\beta}_2}{1000}$$

in other words $\hat{\theta}_2$ = impact of 1 USD ↑ in income

$\hat{\beta}_2$ = impact of 1000 USD ↑ in income

- what if we use bweght in kg & income in THB.

$$bweght_{kg} = \frac{\hat{\beta}_0}{1000} + \frac{\hat{\beta}_1}{1000} cigs + \frac{\left(\frac{\hat{\beta}_2}{1000}\right)}{1000} famin_c$$

This value is going to be 30,000 times more than faminc.

2 More on functional forms

- Logarithmic Functional Form

usually means natural log

$$\log(y) = \beta_0 + \beta_1 \log(x_1) + \beta_2 x_2 + u$$

$\Delta y = y_1 - y_2$

$\Delta x_1 = x_{11} - x_{12}$

$$\beta_1 = \frac{d\log(y)}{d\log(x)} = \frac{\frac{1}{y}dy}{\frac{1}{x_1}dx_1} = \frac{\frac{1}{y}\Delta y}{\frac{1}{x_1}\Delta x_1} = \frac{100 \times \frac{1}{y}\Delta y}{100 \times \frac{1}{x_1}\Delta x_1} = \frac{\% y}{\% x}$$

with the log y & log x format, the coefficient is going to be the elasticity!!!

(x, elasticity of y)

(price) (demand)

$$\beta_2 = \frac{d\log(y)}{dx_2} = \frac{\frac{1}{y}dy}{dx_2} = \frac{\frac{1}{y}\Delta y}{\Delta x_2}$$

» If we want the uppen term to be % change, then

$$100\beta_2 = \frac{100\frac{1}{y}\Delta y}{\Delta x_2}$$

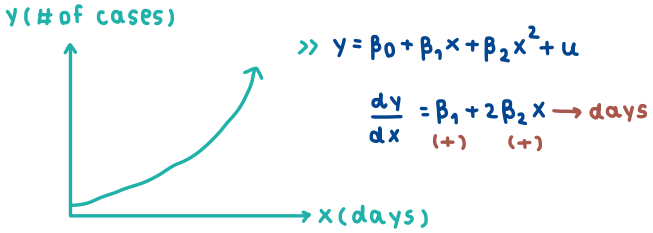
$$100\beta_2 = \frac{\%\Delta y}{\Delta x_2}$$

$\therefore 100\beta_2 = \%$ in y given that x_2 increase by 1 unit.

- Models with Quadratics (squares)

» capture increasing / decreasing marginal effects (slope of the relationship between x & y is not constant)

COVID-19 example

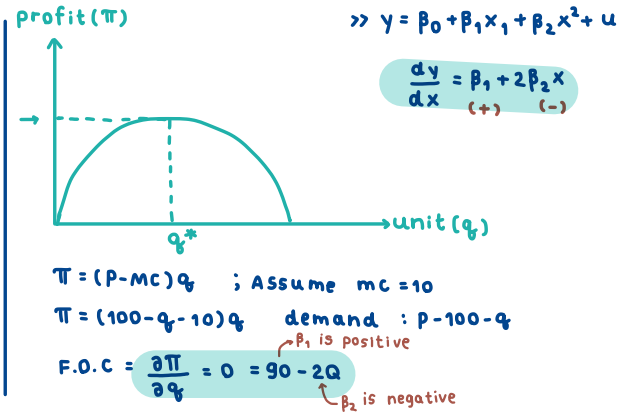


» $y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$

$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$ → days

(+) (+)

Decreasing returns.



» $y = \beta_0 + \beta_1 x_1 + \beta_2 x^2 + u$

$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$

(+) (−)

$\pi = (P - MC)q$; Assume $MC = 10$

$\pi = (100 - q - 10)q$ demand : $P = 100 - q$

F.O.C = $\frac{\partial \pi}{\partial q} = 0 = 90 - 2Q$

β_1 is positive

β_2 is negative

Example : Effects of Pullution on Housing Prices

$$\log(price) = \beta_0 + \beta_1 \log(nox) + \beta_2 \log(dist) + \beta_3 rooms + \beta_4 room^2 + \beta_5 stratio + u$$

where
price = housing price
nox = level of pollution
dist = distance from downtown
rooms = number of rooms
stratio = average student per teacher ratio
The estimation result is given by

In the US or many other countries, students can apply to school in the area w/o having to take any test.
So, the lower stratio, the better school.

```
regress lprice lnox dist rooms rooms_sq stratio
```

Source	SS	df	MS
Model	51.4933152	5	10.298663
Residual	33.0889098	500	.06617782
Total	84.582225	505	.167489554

Number of obs = 506
F(5, 500) = 155.62
Prob > F = 0.0000
R-squared = 0.6088
Adj R-squared = 0.6049
Root MSE = .25725

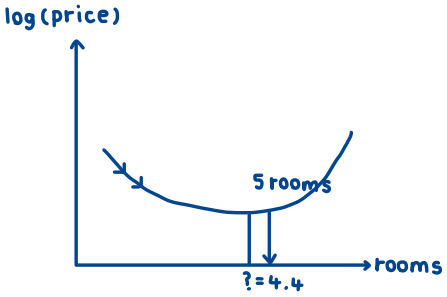
log(price) → lprice, log(nox) → lnox

	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
lnox	β_1 −.9767545	.0995938	−9.81	0.000	−1.172429	−.7810806
dist	β_2 −.0321972	.0094013	−3.42	0.001	−.050668	−.0137264
rooms	β_3 −.5528032	.1612965	−3.43	0.001	−.8697056	−.2359007
rooms_sq	β_4 .0624697	.0124867	5.00	0.000	.0379368	.0870025
stratio	β_5 −.0486667	.0058131	−8.37	0.000	−.0600879	−.0372455
_cons	intc 13.59154	.5650901	24.05	0.000	12.4813	14.70178

|t| > 1.96 >> all variables are significant
all < 0.05

Consider the effect of "room"

$$\frac{d\log(price)}{d\,rooms} = \beta_3 + 2\beta_4 rooms = -0.553 + 2(0.062)\cdot rooms$$



* At how many rooms does 1 additional room has a positive impact on log(price) ??

$0 = -0.553 + 2(0.062)rooms$

$rooms = 4.4$

Ans at 4.4 rooms or more
at 5 rooms or more

What wold be the % change in price when the number of room increases from 5 to 6?

$$\frac{d\log(prices)}{d\,rooms} = -0.553 + 2(0.062)rooms$$

$$100 \times \frac{1}{price}\frac{d\,price}{d\,rooms} = 100 \times 0.067 = 6.7\% \text{ increase}$$

• total %Δ in price when # rooms ↑ from 5 to 7 is 6.7 + 19.1 = 25.8%

>> what about %. in price when # rooms increase from 5 to 7 ??

%Δ price = 100(−0.553 + 2(0.062)·6) = 19.1%

3 Models with Interaction Terms

» used when the impact of one variable depends on the value (level) of another variable.

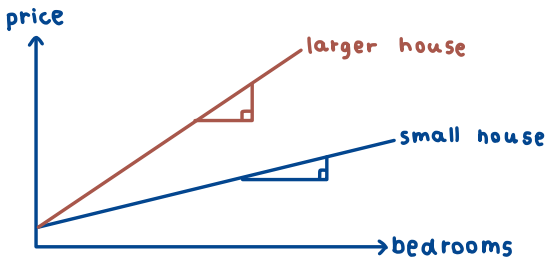
Consider

$$price = \beta_0 + \beta_1 sqrft + \beta_2 bdrms + \beta_3 \underbrace{sqrft \times bdrms}_{x_3} + \beta_4 bthrms + u$$

x_1 x_2 $x_1 \cdot x_2$ x_2

where

$price$ = housing price
$sqrft$ = house size (square feet)
$bdrms$ = number of bedrooms
$bthrms$ = number of bathrooms



$$\frac{\partial \text{ price}}{\partial \text{ bdrms}} = \beta_2 + \beta_3 \text{sqrft}$$

» if $\beta_2 > 0$ then, an additional bedroom would increase price more for a larger house!!

4 More on the Goodness-of-Fit and Selection of Regressors

- Adding more regressors ALWAYS improve fit » R^2 always ↑

• But we lose the "degree of freedom"
(d.f. = free data point used to estimate the parameter)
» 1 data point is sacrificed everytime we estimate a parameter
• using R^2 would not punish "having too many regressors"
• we use adjusted-R^2 or R^{-2} when we want to punish adding too many regressors.

$$R^2 = 1 - \frac{SSR}{SST} = 1 - \frac{\frac{SSR}{n}}{\frac{SST}{n}}$$

$$\text{adj. } R^2 = \frac{1 - \frac{SSR}{(n-k-1)}}{\frac{SST}{(n-1)}}$$

Using adjusted R-squared to choose between non-nested models (one model is not a subset of another).

Consider Model 1

If we have more k, d.f. = n-k-1 ↓, $\frac{SSR}{(n-k-1)}$ ↑, adj. R^2 ↓

$$\begin{aligned} \widehat{salary} &= \underset{(223.90)}{830.63} + \underset{(0.0089)}{0.0163}sales + \underset{(11.08)}{19.63}roe \\ n &= 209, \quad R^2 = 0.029, \quad \bar{R}^2 = 0.020 \end{aligned}$$

Consider Model 2

$$\begin{aligned} \widehat{\log(salary)} &= \underset{(0.29)}{4.36} + \underset{(0.033)}{0.2751}\log(sales) + \underset{(0.004)}{0.0179}roe \\ n &= 209, \quad R^2 = 0.282, \quad \bar{R}^2 = 0.275 \end{aligned}$$

» 27.5% of variation in y is explained. So, model is better!!

Multiple Regression Analysis with Qualitative Information:

1 Outline

- Describing qualitative information
- Using a single dummy independent variable
- Using dummy variables for multiple categories
- Interactions involving dummy variables
- A binary dependent variable (Y variable): The linear probability model

2 Describing Qualitative Information

- "Female" and "Married" are qualitative variable.
- We arbitarily assign a dummy variable to decribe them.

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

$$married = \begin{cases} 1 & \text{if married} \\ 0 & \text{otherwise (of if single)} \end{cases}$$

TABLE 7.1

A Partial Listing of the Data in WAGE1.RAW

person	wage	educ	exper	female	married
1	3.10	11	2	1	0
2	3.24	12	22	1	1
3	3.00	11	2	0	0
4	6.00	8	44	0	1
5	5.30	12	7	0	1
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
525	11.56	16	5	0	1
526	3.50	14	5	1	0

3 Models with a single dummy independent variable

Consider

$$wage = \beta_0 + \delta_0 female + \beta_1 educ + u.$$

where

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

In this case, the δ_0 notation is used to highlight the interpretation of the parameters multiplying dummy variables. In other cases, we can use any notation that is the most convenient.

1) $E(wage \mid female, edu) = E(\beta_0 + \delta_0 female + \beta_1 edu + u \mid female, edu)$

$= \beta_0 + \delta_0 female + \beta_1 edu + E(u \mid female, edu)$

$= \beta_0 + \delta_0 female + \beta_1 edu$ $\quad \downarrow = 0$ (ass MLR1-4)

2) Thus

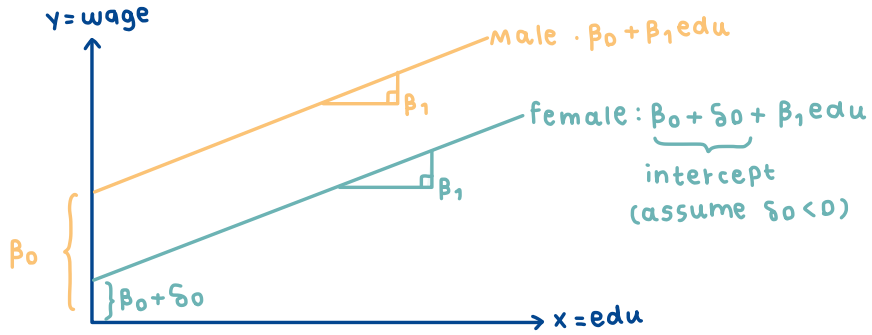
♀ : $E(wage \mid female = 1, edu) = \beta_0 + \delta_0 (1) + \beta_1 edu = \beta_0 + \delta_0 + \beta_1 edu$

♂ : $E(wage \mid female = 0, edu) = \beta_0 + \delta_0 (0) + \beta_1 edu = \beta_0 + \beta_1 edu$

$\delta_0 = E(wage \mid female = 1, edu) - E(wage \mid female = 0, edu)$

OR $\delta_0 = E(wage \mid female, edu) - E(wage \mid male, edu)$

* given that same value of edu (same education level), δ_0 is the difference in the expected wage of females and males.



>> By the way we model this regression function "female" is going to give a constant impact on wage. Regardless of the level of edu

4 It is not possible to include all of the dummy alternatives in the same model (as long as there is an intercept in the model)

- If we include all alternatives of a dummy variable in the same model, we will face the "perfect collinearity" problem.

For example:

$$wage = \beta_0 x_0 + \delta_0 female + \beta_1 edu + \delta_1 male + u$$

(x_0: intercept = 1; x_1; x_2; x_3)

$$x_0 = x_1 + x_3$$

$$1 = female + male$$
$$female = male + 1$$

id.	fem	male	x_0
1	1	0	1
2	1	0	1
3	0	1	1
4	0	1	1
.	0	1	1
.	0	1	1
.	1	0	1
.	1	0	1
.			
.			
99			

or

If there are "n" catagories, we omit '1' category avoid multi collinearity

$$1 = winter + spring + summer + fall \quad \checkmark\checkmark$$
$$winter = 1 - spring - summer - fall$$

$$winter = \begin{cases} 1 & \text{if winter} \\ 0 & \text{otherwise} \end{cases}$$

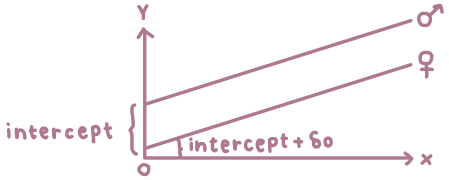
$$spring = \begin{cases} 1 & \text{if spring} \\ 0 & \text{otherwise} \end{cases}$$

etc.

id	winter	spring	summer	fall	x_0
1	1	0	0	0	1
2	1	0	0	0	1
3	0	0	1	0	1
4	0	0	1	0	1
.	0	1	0	0	1
.	0	1	0	0	1

- At least one alternative has to be dropped. We treat the dropped alternative as the "BASE GROUP" or "BASELINE" or "BENCHMARK GROUP". ↳ In this case, male

1 if female, 0 if male — 1 if male, 0 if female




```
. regress lwage female male married educ exper
note: male omitted because of collinearity

      Source |       SS       df       MS              Number of obs =     526
-------------+------------------------------           F(  4,   521) =   75.27
       Model |  54.3265253     4  13.5816313            Prob > F      =  0.0000
    Residual |  94.0032262   521  .180428457           R-squared     =  0.3663
-------------+------------------------------           Adj R-squared =  0.3614
       Total |  148.329751   525  .28253286            Root MSE      =  .42477

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      female |  -.3251146   .0377061    -8.62   0.000    -.3991892    -.25104
        male |           0  (omitted)
     married |   .1380145   .0411197     3.36   0.001     .0572338    .2187953
        educ |   .0872644   .0071554    12.20   0.000     .0732075    .1013213
       exper |   .0076213   .0015314     4.98   0.000     .0046129    .0106297
       _cons |   .4690918   .1040575     4.51   0.000      .264668    .6735156
------------------------------------------------------------------------------
```

Female workers are expected to have less wage compared to male workers.

5 Using dummy variables for multiple categories

Case 1 We can use many dummy variables in the same model

Consider a model which includes 2 dummy variables– *female* and *married*.

$$\log(wage) = \beta_0 + \delta_0 female + \delta_1 married + \beta_1 educ + \beta_2 \exp er + \beta_3 \exp er^2 + \beta_4 tenure + \beta_5 tenure^2 + u.$$

1 if female, 0 if male

1 if female, 0 if male

```
regress lwage female married educ exper expersq tenure tenursq
```

Source	SS	df	MS
Model	65.6482326	7	9.37831895
Residual	82.6815188	518	.159616832
Total	148.329751	525	.28253286

Number of obs	= 526
F(7, 518)	= 58.76
Prob > F	= 0.0000
R-squared	= 0.4426
Adj R-squared	= 0.4351
Root MSE	= .39952

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.2901838	.0361121	-8.04	0.000	-.3611279	-.2192396
married	.0529219	.0407561	1.30	0.195	-.0271456	.1329894
educ	.0791547	.0068003	11.64	0.000	.0657952	.0925143
exper	.0269535	.0053258	5.06	0.000	.0164907	.0374163
expersq	-.0005399	.0001122	-4.81	0.000	-.0007603	-.0003196
tenure	.0312962	.0068482	4.57	0.000	.0178426	.0447499
tenursq	-.0005744	.0002347	-2.45	0.015	-.0010355	-.0001134
_cons	.4177837	.0988662	4.23	0.000	.2235557	.6120116

Comments:

1) δ_0 measures the expected difference between female & male workers given the same marital status and other factors.

$$\frac{\partial \log(wage)}{\partial female} = \frac{\frac{1}{wage}\, d\,wage}{\partial female} = -0.29$$

$$= \frac{100 \cdot \frac{1}{wage}\, d\,wage}{\partial female} = 100(-0.29)$$

$$= \frac{\%\,\Delta wage}{\partial female} = 29.02\%$$

- female workers are expected to earn less than male workers by 29.02%, holding other factors the same

δ_1 measure the impact of be married (marriage premium)

But since $|t| < 1.96$ or $p > 0.05$, we do not reject H_0 of no impact.

	♀	♂
marr	marrfem	marrmale
sing	singfem	singmale (base case)

Consider a model which includes dummy variables for each gender/marital status combination– *marrmale*, *marrfem* and *singfem*. (or singmale ← used the basecase)

$$\log(wage) = \beta_0 + \delta_0 marrmale + \delta_1 marrfem + \delta_3 singfem + \beta_1 educ + \beta_2 exper + \beta_3 exper^2 + \beta_4 tenure + \beta_5 tenure^2 + u. \quad (8.1)$$

```
regress lwage marrmale marrfem singfem educ exper expersq tenure tenursq
```

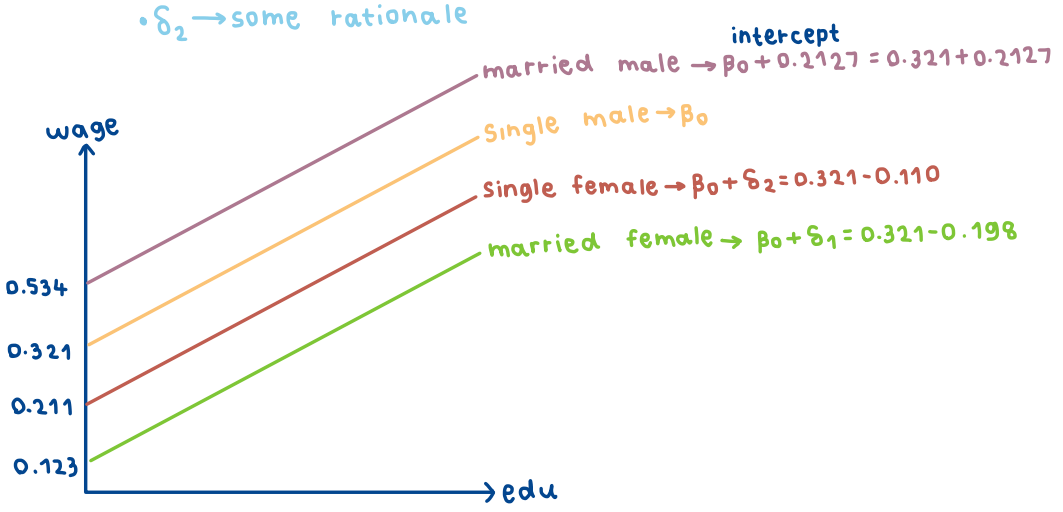
Source	SS	df	MS
Model	68.3617623	8	8.54522029
Residual	79.9679891	517	.154676961
Total	148.329751	525	.28253286

Number of obs	= 526
F(8, 517)	= 55.25
Prob > F	= 0.0000
R-squared	= 0.4609
Adj R-squared	= 0.4525
Root MSE	= .39329

	lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
δ_0	marrmale	.2126757	.0553572	3.84	0.000	.103923	.3214284
δ_1	marrfem	-.1982676	.0578355	-3.43	0.001	-.311889	-.0846462
δ_2	singfem	-.1103502	.0557421	-1.98	0.048	-.219859	-.0008414
$\hat{\beta}$	educ	.0789103	.0066945	11.79	0.000	.0657585	.092062
	exper	.0268006	.0052428	5.11	0.000	.0165007	.0371005
	expersq	-.0005352	.0001104	-4.85	0.000	-.0007522	-.0003183
	tenure	.0290875	.006762	4.30	0.000	.0158031	.0423719
	tenursq	-.0005331	.0002312	-2.31	0.022	-.0009874	-.0000789
	_cons	.3213781	.100009	3.21	0.001	.1249041	.5178521

Comments: This regression is not the same as the previous one. It uses "single male" as the base group. (The previous one use male & single as 2 base groups).

- δ_0 measures the expected diff. in wage of married male as compare with single males, holding other factors constant.
- δ_1 measures the expected diff. in wage of married female as compared with single males, holding other factors constant.
- δ_2 → some rationale



Case 2 We can use dummy variables to represent multiple categories of a variable Consider the relationship between law school rankings and starting salaries

$$\log(salary) = \beta_0 + \delta_0 top10 + \delta_1 r11\_25 + \delta_3 r26\_40 + \delta_4 r41\_60 + \beta_1 LSAT + \beta_2 GPA + \beta_3 \log(libvol) + \beta_4 \log(cost) + u.$$

where $top10$, $r11\_25$, $r26\_40$, $r41\_60$ would be equal to 1 when the variable $rank$ falls into the appropriate range.

** Rank below 60 would be the base case.

In many case the "range of the value" serve as a better explanatory variable than the "value it self" eg. age may explain the model better if split into generations young 0-15 genz 16-29 etc.

```
. regress lsalary top10 r11_25 r26_40 r41_60 LSAT GPA llibvol lcost
```

Source	SS	df	MS		
Model	9.16538532	8	1.14567316	Number of obs	= 136
Residual	1.2109665	127	.009535169	F(8, 127)	= 120.15
				Prob > F	= 0.0000
				R-squared	= 0.8833
				Adj R-squared	= 0.8759
Total	10.3763518	135	.076861865	Root MSE	= .09765

lsalary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
top10	.5393428	.053542	10.07	0.000	.4333927	.6452928
r11_25	.4716199	.0390921	12.06	0.000	.3942637	.548976
r26_40	.2790977	.0346972	8.04	0.000	.2104383	.3477571
r41_60	.182382	.0283098	6.44	0.000	.126362	.238402
LSAT	.0060482	.0034919	1.73	0.086	-.0008616	.012958
GPA	.1305893	.0818678	1.60	0.113	-.0314122	.2925908
llibvol	.0725522	.0289213	2.51	0.013	.0153221	.1297824
lcost	.0249169	.0283224	0.88	0.381	-.031128	.0809619
_cons	8.363103	.4457314	18.76	0.000	7.481081	9.245125

the baseline is ranking 61th and worse.

Comments:

rank	top 10	r11-25	r26-40
1	1	0	0
2	1	0	0
3	1	0	0
⋮	1	0	⋮
10	1	0	⋮
11	0	0	⋮
12	0	1	⋮
⋮	0	1	0
	0	1	0
25	0	1	0
26	0	1	0
⋮		0	1
		0	1
		0	1
40			1

1) δ_0 measures the difference in expected log(salary) of a law-school graduate from top 10 university compared to expected log(salary) of those who graduate from the school ranked 61th and worse.

2) δ_1 → use the same rationale.