Solution: Practice Problems - Final Exam

1. Let a and b be real numbers with $0 < b < a < 1$. Determine whether each of the following inequalities is true or not. Explain your answer.

   (a) $\frac{(b+1)^3}{|a|} < \frac{(a+1)^3}{b}$

   (b) $\frac{b}{a} < \frac{a^3-1}{b^3-1}$

   Answer:

   (a) $\frac{(b+1)^3}{|a|} < \frac{(a+1)^3}{b}$ is **true**.
   From $0 < b < a < 1$, by adding 1 throughout the inequalities, we have

   $$0+1 < b+1 < a+1 < 1+1$$

   and so, $0 < b+1 < a+1$

   $$0 < (b+1)^3 < (a+1)^3. \tag{1}$$

   Since $a > 0$, $|a| = a$. Since $a, b > 0$, then $b < a$ implies $\frac{1}{ab}b < \frac{1}{ab}a$ or

   $$0 < \frac{1}{|a|} < \frac{1}{b} \tag{2}$$

   From (1) and (2),

   $$\frac{(b+1)^3}{|a|} < \frac{(a+1)^3}{b}.$$

   (b) $\frac{b}{a} < \frac{a^3-1}{b^3-1}$ is **false**.

   Since $a, b > 0$, $b < a$ implies

   $$\frac{b}{a} < 1.$$

   From $0 < b < a < 1$, we have $0 < b^3 < a^3 < 1$ and by subtracting 1 throughout the inequalities, we have

   $$-1 < b^3 - 1 < a^3 - 1 < 0.$$

   Since $b^3 - 1 < 0$, when we divide $b^3 - 1$ throughout the equality $b^3 - 1 < a^3 - 1$ we have

   $$\frac{a^3-1}{b^3-1} < 1.$$

   That is, we only have $\frac{b}{a} < 1$ and $\frac{a^3-1}{b^3-1} < 1$ and we cannot conclude that $\frac{b}{a} < \frac{a^3-1}{b^3-1}$.
   A counterexample is when $a = \frac{4}{5}$ and $b = \frac{3}{4}$. We noticed that $0 < \frac{3}{4} < \frac{4}{5} < 1$ and so $0 < b < a < 1$,

   $$\frac{b}{a} = \frac{3/4}{4/5} = \frac{15}{16} = 0.9375$$

   and

   $$\frac{a^3-1}{b^3-1} = \frac{(4/5)^3-1}{(3/4)^3-1} = 2063/2444 \approx 0.8441$$

   Since $0.9375 \not< 0.844$, we can find a and b with $0 < b < a < 1$ such that $\frac{b}{a} \not< \frac{a^3-1}{b^3-1}$. ■

2. Let x and y be real numbers with $y > x > 1$. Show that

$$y(y-2) > |x-1|^2 - 1.$$

Answer: From $y > x > 1$, subtracting 1 throughout these inequalities gives

$$y - 1 > x - 1 > 0,$$

and since both $y-1$ and $x-1$ are positive numbers, we can square both sides of $y-1 > x-1$:

$$(y-1)^2 > (x-1)^2$$

or equivalently,

$$y^2 - 2y + 1 > (x-1)^2 \quad \Leftrightarrow \quad y^2 - 2y > (x-1)^2 - 1 \quad \Leftrightarrow \quad y(y-2) > (x-1)^2 - 1$$

Since $(x-1)^2 = |x-1|^2 \ \forall x \in \mathbb{R}$, then we have

$$y(y-2) > |x-1|^2 - 1.$$

■

3. Find the solution set for each of following inequalities.

(a)

$$\frac{x^2-1}{x} < \frac{x+1}{2x}$$

(b)

$$2 \leq \left| \frac{x-1}{x} \right| \leq 7$$

(c)

$$\frac{x^2+|x|+1}{x^7+x^5+x^2+1} \leq 0$$

(d)

$$\frac{|x+3|-2}{5} + \frac{1}{|x-1|+1} \leq 1$$

(e)

$$\frac{|x-1|-x^2-1}{5-|x+3|} \geq 0$$

Answer:

(a)

$$\frac{x^2-1}{x} < \frac{x+1}{2x}$$

Answer:

$$\begin{aligned}
\frac{x^2-1}{x} &< \frac{x+1}{2x}\\
\frac{x^2-1}{x}-\frac{x+1}{2x} &< 0\\
\frac{2x^2-2-x-1}{x} &< 0\\
\frac{2x^2-x-3}{x} &< 0\\
\frac{(x+1)(2x-3)}{x} &< 0
\end{aligned}$$

Notice that $(x+1)(2x-3) =$ when $x = -1, 3/2$ and $x = 0$ when $x = 0$. To obtain the subintervals, consider $x = -1, 0, 3/2$

	$x \in (-\infty,-1)$	$x \in (-1,0)$	$x \in (-1,3/2)$	$x \in (3/2,\infty)$
$x+1$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$2x-3$	$-$	$-$	$-$	$+$
$\frac{(x+1)(2x-3)}{x}$	$\boxed{-}$	$+$	$\boxed{-}$	$+$

To have non-zero denominator, we must have $x \neq 0$. Since we consider $<$ sign, the solution set is $(-\infty,-1) \cup (-1,3/2)$. ■

(b)

$$2 \leq \left|\frac{x-1}{x}\right| \leq 7$$

Answer: This is true when $2 \leq \left|\frac{x-1}{x}\right|$ and $\left|\frac{x-1}{x}\right| \leq 7$.
(i) Consider $2 \leq \left|\frac{x-1}{x}\right|$. This is equivalent to

$$\begin{aligned}
|2|^2 &\leq \left|\frac{x-1}{x}\right|^2\\
4 &\leq \frac{(x-1)^2}{x^2} \qquad \text{since } \left|\frac{x-1}{x}\right|^2 = \frac{|x-1|^2}{|x|^2} = \frac{(x-1)^2}{x^2}\\
4x^2 &\leq (x-1)^2, \qquad \text{since } x^2 > 0\\
4x^2 &\leq x^2-2x+1\\
3x^2+2x-1 &\leq 0\\
(x+1)(3x-1) &\leq 0.
\end{aligned}$$

By setting $(x+1)(3x-1) = 0$, we consider $x = -1$ and $x = 1/3$ to construct 3 subintervales.

	$x \in (-\infty,-1)$	$x \in (-1,1/3)$	$x \in (1/3,\infty)$
$x+1$	$-$	$+$	$+$
$3x-1$	$-$	$-$	$+$
$(x+1)(3x-1)$	$+$	$\boxed{-}$	$+$

So, in this case, since $|x|$ is the denominator, $x \neq 0$ and the solution set is $\in [-1, 1/3] - \{0\}$ or $[-1, 0) \cup (0, 1/3]$

(ii) Consider $\left|\frac{x-1}{x}\right| \leq 7$ This is equivalent to

$$\begin{aligned}
\left|\frac{x-1}{x}\right|^2 &\leq |7|^2 \\
\frac{(x-1)^2}{x^2} &\leq 49 \qquad \text{since } \left|\frac{x-1}{x}\right|^2 = \frac{|x-1|^2}{|x|^2} = \frac{(x-1)^2}{x^2} \\
(x-1)^2 &\leq 49x^2, \qquad \text{since } x^2 > 0 \\
49x^2 &\geq x^2 - 2x + 1 \\
48x^2 + 2x - 1 &\geq 0 \\
(6x+1)(8x-1) &\geq 0.
\end{aligned}$$

By setting $(6x+1)(8x-1) = 0$, we consider $x = -1/6$ and $x = 1/8$ to construct 3 subintervales.

	$x \in (-\infty, -1/6)$	$x \in (-1/6, 1/8)$	$x \in (1/8, \infty)$
$(6x+1)$	$-$	$+$	$+$
$(8x-1)$	$-$	$-$	$+$
$(6x+1)(8x-1)$	$\boxed{+}$	$-$	$\boxed{+}$

So, in this case, since $|x|$ is the denominator, $x \neq 0$ and the solution set is $(-\infty, -1/6] \cup [1/8, \infty)$.

From (i) and (ii), the solution set is
$\{[-1, 0) \cup (0, 1/3]\} \cap \{(-\infty, -1/6] \cup [1/8, \infty)\} = [-1, -1/6] \cup [1/8, 1/3]$. ■

(c)

$$\frac{x^2 + |x| + 1}{x^7 + x^5 + x^2 + 1} \leq 0$$

Answer:
First notice that

$$x^7 + x^5 + x^2 + 1 = (x^7 + x^5) + (x^2 + 1) = x^5(x^2 + 1) + (x^2 + 1) = (x^5 + 1)(x^2 + 1).$$

Since $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$, we will consider two cases for x.

Case I: $x < 0$. Then $|x| = -x$ and

$$x^2 + |x| + 1 = x^2 - x + 1.$$

Notice that
$x^2 - x + 1 > 0$ for all $x \in \mathbb{R}$ ($b^2 - 4ac = 1 - 4(1)(1) < 0$).

Case II: $x \geq 0$. Then $|x| = x$ and

$$x^2 + |x| + 1 = x^2 + x + 1.$$

Notice that
$x^2+x+1>0$ for all $x\in\mathbb{R}$, $(b^2-4ac=1-4(1)(1)<0)$.
From cases (I) and (II), we have that the numerator

$$x^2+|x|+1>0$$

for all $x\in\mathbb{R}$. Not also that, since $x^2\geq 0$ and $x^2+1\geq 1>0$,

$$x^2+1>0, \qquad x\in\mathbb{R},$$

so we have that

$$\frac{x^2+|x|+1}{(x^5+1)(x^2+1)}<0$$

only when

$$\frac{1}{(x^5+1)}<0 \qquad \text{or } x^5+1<0$$

Note that $x^5+1=0$ when $x^5=-1$ or $x=\sqrt[5]{-1}=-1$. So we consider the intervals $(-\infty,-1)$ and $(-1,\infty)$.

	$x\in(-\infty,-1)$	$x\in(-1,\infty)$
x^5+1	$\boxed{-}$	$+$

From above, the solution set is therefore $(-\infty,-1)$ ■

(d)

$$\frac{|x+3|-2}{5}+\frac{1}{|x-1|+1}\leq 1$$

(e)

$$\frac{|x-1|-x^2-1}{5-|x+3|}\geq 0$$

Answer:
From $|x-1|=\begin{cases}-(x-1), & x-1<0 \Leftrightarrow x<1\\ x-1, & x-1\geq 0 \Leftrightarrow x\geq 1\end{cases}$, and $|x+3|=\begin{cases}-(x+3), & x+3<0 \Leftrightarrow x<-3\\ x+3, & x+3\geq 0 \Leftrightarrow x\geq -3\end{cases}$,
we will consider 3 cases:

Case I: $x\in(-\infty,-3)$, Case II:$x\in[-3,1)$, Case III: $x\in[1,\infty)$.

Case I: $x\in(-\infty,-3)$, $|x-1|=-(x-1)$, $|x+3|=-(x+3)$ and

$$\frac{x^2+1-|x-1|}{5-|x+3|}=\frac{x^2+1+x-1}{5+x+3}=\frac{x^2+x}{x+8}=\frac{x(x+1)}{x+8}.$$

So we will solve $\frac{x(x+1)}{x+8}\leq 0$. To obtain the subintervals, consider $x=-8,-1,0$

	$x\in(-\infty,-8)$	$x\in(-8,-1)$	$x\in(-1,0)$	$x\in(0,\infty)$
$x+8$	$-$	$+$	$+$	$+$
$x+1$	$-$	$-$	$+$	$+$
x	$-$	$-$	$-$	$+$
$\frac{x(x+1)}{x+8}$	$\boxed{-}$	$+$	$\boxed{-}$	$+$

To have non-zero denominator, we must have $x \neq -8$. Since we have "less than or equal" sign, then the solution set is $(-\infty,-8) \cup [-1,0]$. I.e., the solution set is $(-\infty,-3) \cap \{(-\infty,-8) \cup [-1,0]\} = \boxed{(-\infty,-8)}$.

Case II: $x \in [-3,1)$, $|x-1| = -(x-1)$, $|x+3| = x+3$ and

$$\frac{x^2+1-|x-1|}{5-|x+3|} = \frac{x^2+1+x-1}{5-x-3} = \frac{x^2+x}{-x+2} = -\frac{x(x+1)}{x-2}.$$

So we will solve $-\frac{x(x+1)}{x-2} \leq 0$ or $\frac{x(x+1)}{x-2} \geq 0$. To obtain the subintervals, consider $x = -1, 0, 2$

	$x \in (-\infty,-1)$	$x \in (-1,0)$	$x \in (-1,2)$	$x \in (2,\infty)$
$x+1$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$\frac{x(x+1)}{x-2}$	$-$	$\boxed{+}$	$-$	$\boxed{+}$

To have non-zero denominator, we must have $x \neq 2$. Since we have "less than or equal" sign, then $x \in [-1,0] \cup (2,\infty)$. I.e., the solution set is $[-3,1) \cap \{x \in [-1,0] \cup (2,\infty)\} = \boxed{[-1,0]}$.

Case III: $x \in [1,\infty)$, $|x-1| = x-1$, $|x+3| = x+3$ and

$$\frac{x^2+1-|x-1|}{5-|x+3|} = \frac{x^2+1-x+1}{5-x-3} = \frac{x^2-x+1}{-x+2} = -\frac{x^2-x+1}{x-2}.$$

So we want to solve

$$\frac{x^2-x+1}{x-2} \geq 0.$$

For $a=1, b=-1, c=1$, $b^2-4ac = 1-4 < 0$, so we cannot factor x^2-x+1. Since $a > 0$, then $x^2-x+1 > 0$ for all $x \in \mathbb{R}$. So, in order to have $\frac{x^2-x+1}{x-2} \geq 0$, we can consider instead $x-2 > 0$ or $x \in (2,\infty)$. That is, the solution set for this case is $[1,\infty) \cap (2,\infty) = \boxed{(2,\infty)}$.

From cases I, II, III, the solution set is $(-\infty,-8) \cup [-1,0] \cup (2,\infty)$. ■

4. Let $X = \{1,2,3\}$ and $Y = \{a,b,c,d\}$. Define relations
$f : X \to Y$ by $f = \{(1,a),(2,a),(3,c)\}$,
$g : X \to Y$ by $g = \{(1,a),(3,c)\}$, and
$h : X \to Y$ by $h = \{(1,a),(2,a),(3,b),(3,c)\}$.

(a) Draw the arrow diagrams of f, g, and h.

(b) Show that f is a function, but g and h are not functions.

(c) Find the domain of f, co-domain of f,and range of f.

(d) What is the inverse image of a for the function f?

(e) What is $f(3)$?

Answer:

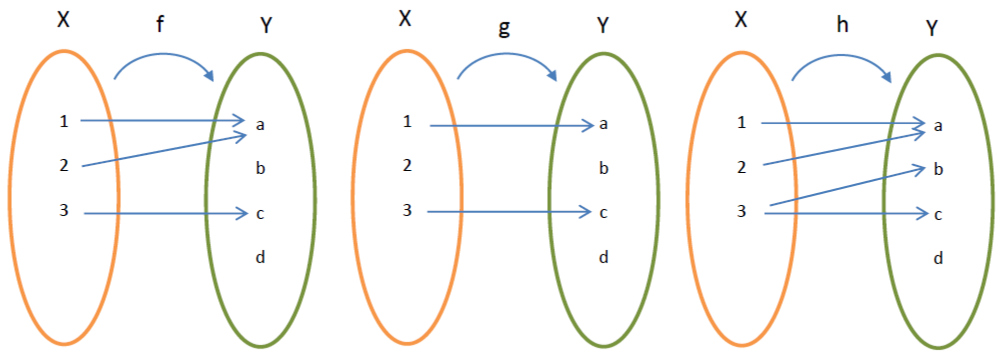

Figure 1: Problem 1(a)

(a) Draw the arrow diagrams of f, g, and h.

(b) Show that f is a function, but g and h are not functions.
f is a function because(i) every element in the domain is used and (ii) each $x \in X$ gets mapped exactly once(for each $x \in X$, there is a unique $y \in Y$ so that $(x, y) \in f$, or if $(x, y_1), (x, y_2) \in f$ then $y_1 = y_2$).

(c) Find the domain of f, co-domain of f,and range of f.
The domain is $X = \{1, 2, 3\}$
The co-domain is $Y = \{a, b, c, d\}$.
The range is $\Big\{a, c\Big\}$.

(d) What is the inverse image of a for the function f?
The inverse image of a are the elements in X that get mapped to a:

$$\text{Inverse Image of } a = \{1, 2\}.$$

(e) What is $f(3)$?
Since $(3, c) \in f$, then $f(3) = c$.

5. Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For all $(x, y) \in A \times B$,

$$x \ R \ y \Leftrightarrow \ |x| = |y|.$$

and

$$x \ S \ y \ \Leftrightarrow \ x - y \text{ is even.}$$

State explicitly the sets $A \times B$, R, S, $R \cup S$, and $R \cap S$.

Answer:

- $A \times B = \{(x, y)|x \in A, y \in B\} = \{(-1, 1), (1, 1), (2, 1), (4, 1), (-1, 2), (1, 2), (2, 2), (4, 2)\}$
- $R = \{(x, y) \in A \times B||x| = |y|\} = \{(-1, 1), (1, 1), (2, 2)\}$

- $R = \{(x, y) \in A \times B | x - y \text{ is even}\} = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$
- $R \cup S = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$
- $R \cap S = \{(-1, 1), (1, 1), (2, 2)\}$

6. Let $A = \{1, 2, 3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A, and
$$X = \{x \in \mathcal{P}(A) |\ x \cap \{1\} \neq \emptyset\}.$$
Define a relation r from X to $\mathbb{Z}$ as
$$r = \{(x, y) \in X \times \mathbb{Z} |\ y = \text{ the number of elements in } x\}.$$
   (a) List all elements in X.
   (b) Draw an arrow diagram of r .
   (c) Is r a function? If so, find the domain, co-domain, and range of r.

Answer:

   (a) List all elements in X.
   Since $\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$,
$$X = \{x \in \mathcal{P}(A) |\ x \cap \{1\} \neq \emptyset\} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}.$$
   (b) Draw an arrow diagram of r .

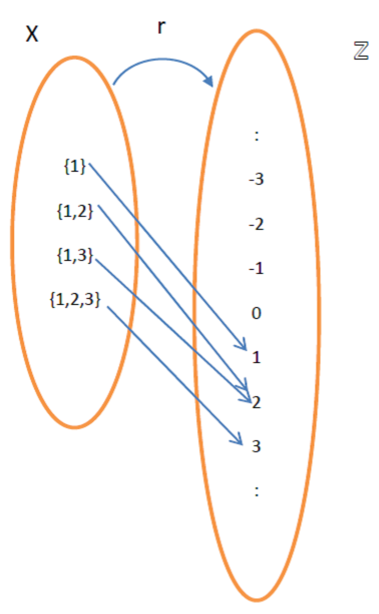

Figure 2: Problem 3(b)

(c) Is r a function? If so, find the domain, co-domain, and range of r.
Yes, r is a function because(i) every element in the domain is used and (ii) for each $x \in X$, there is a unique $y \in \mathbb{Z}$ so that $(x, y) \in r$, or if $(x, y_1), (x, y_2) \in r$ then $y_1 = y_2$ (i.e. each $x \in X$ gets mapped exactly once).
The domain is $X = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$
The co-domain is $\mathbb{Z} = \Big\{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \Big\}$.
The range is $\Big\{1, 2, 3\Big\}$.

7. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (x + 1, 2y - 3) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

(a) Is H one-to-one? Prove or give a counterexample.
(b) Is H onto? Prove or give a counterexample.
(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

<u>Answer:</u>

(a) Yes, H is one-to-one. To prove this, we will show that for (x_1, y_1) and (x_2, y_2) in the domain, if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$. Suppose $H(x_1, y_1) = H(x_2, y_2)$, Then

$$\begin{aligned} H(x_1, y_1) &= H(x_2, y_2) \\ (x_1 + 1, 2y_1 - 3) &= (x_1 + 1, 2y_1 - 3) \end{aligned}$$

and this is equivalent to $x_1 + 1 = x_2 + 1 \quad$ and $\quad 2y_1 - 3 = 2y_2 - 3$, or

$$x_1 + 1 = x_2 + 1 \quad \Leftrightarrow \quad x_1 = x_2$$

$$2y_1 - 3 = 2y_2 - 3 \quad \Leftrightarrow \quad 2y_1 = 2y_2 \quad \Leftrightarrow \quad y_1 = y_2$$

i.e., $(x_1, y_1) = (x_2, y_2)$. That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$. ■

(b) Yes, H is onto. To prove this, we will show that for any (u, v) in the co-domain $\mathbb{R} \times \mathbb{R}$, there exists (x, y) from the domain $\mathbb{R} \times \mathbb{R}$ such that $H(x, y) = (u, v)$.
Suppose, temporarily that, there is (x, y) such that $H(x, y) = (u, v)$. Then,

$$\begin{aligned} H(x, y) &= (u, v) \\ (x + 1, 2y - 3) &= (u, v) \end{aligned}$$

i.e., we must have $x + 1 = u \quad \Leftrightarrow \quad \boxed{x = u - 1}$ and $2y - 3 = v \quad \Leftrightarrow \quad \boxed{y = \frac{v+3}{2}}$.
Since $u, v \in \mathbb{R}$, then $x = u - 1, y = \frac{v+3}{2} \in \mathbb{R}$.
That is, for *any* given (u, v) in the co-domain $\mathbb{R} \times \mathbb{R}$, we can find $(x, y) = \left(u - 1, \frac{v+3}{2}\right) \in \mathbb{R} \times \mathbb{R}$ in the domain, such that

$$H(x, y) = H\left(u - 1, \frac{v + 3}{2}\right) = H\left(u - 1 + 1, 2(\frac{v + 3}{2}) - 3\right) = (u, v + 3 - 3) = (u, v),$$

and hence H is onto. ■

(c) From (a) and (b), since H is both one-to-one and onto, then H bijective.
To find the inverse function, H^{-1}, of H, we recall from the definition

$$H^{-1}(u,v) = (x,y) \qquad \Leftrightarrow \qquad H(x,y) = (u,y).$$

Since we have from (b) that

$$H\left(u-1, \frac{v+3}{2}\right) = (u,v),$$

and hence the inverse function $H^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is given by

$$H^{-1}(u,v) = \left(u-1, \frac{v+3}{2}\right).$$

■

8. Define functions $f_1 : [0, 2) \to \mathbb{R}$ as
$$f_1(x) = x^2$$
and define $f_2 : [2, \infty) \to \mathbb{R}$ as
$$f_2(x) = 3x - 2.$$
Let $F : [0, \infty) \to [0, \infty)$ be a function defined by using f_1 and f_2:
$F(x) = f_1(x)$, for $x \in [0, 2)$, and $F(x) = f_2(x)$, for $x \in [2, \infty)$. That is,
$$F(x) = \begin{cases} x^2, & x \in [0, 2) \\ 3x - 2, & x \in [2, \infty). \end{cases}$$
   (a) Find the domain, co-domain, and range for each of the functions f_1, f_2, and F.
   (b) Construct the composite functions $f_1 \circ f_2$, $f_2 \circ f_1$, and $f_1 \circ F$ (if possible). Determine the domains and ranges for these composite functions.
   (c) Are f_1 and f_2 injective? Explain.
   (d) Is the function F bijective? If so, find the **inverse function** of F.

Answer

(a) The domain, co-domain, and range for each of the functions f_1, f_2, and F.
The domain of f_1 is $D_{f_1} = [0, 2)$, and the range is $R_{f_1} = [0, 4)$, which comes from:
$$0 \le x < 2 \quad \Rightarrow \quad 0 \le x^2 < 4 \quad \Rightarrow \quad 0 \le f_1(x) < 4.$$
The domain of f_2 is $D_{f_2} = [2, \infty)$, and the range is $R_{f_2} = [4, \infty)$, which comes from:
$$2 \le x \quad \Rightarrow \quad 3x - 2 \ge 4 \quad \Rightarrow \quad f(x_2) \ge 4.$$
The domain of F is $D_F = (\infty, 2) \cup (2, \infty) = [0, \infty)$, and the range is $R_F = [0, \infty)$, which comes from:
$$\text{For } x =\in [0, 2), 0 \le x < 2 \quad \Rightarrow \quad 0 \le x^2 < 4 \quad \Rightarrow \quad F(x) \in [0, 4).$$
$$\text{For } x =\in [2, \infty) \quad \Rightarrow \quad 2 \le x \quad \Rightarrow \quad 3x - 2 \ge 4 \quad \Rightarrow \quad F(x) \in [4, \infty).$$

(b) Construct the composite functions $f_1 \circ f_2$, $f_2 \circ f_1$, and $f_1 \circ F$ (if possible). Determine the domains and ranges for these composite functions.

$\underline{f_1 \circ f_2}$ can be defined only when $D_{f_1} \cap R_{f_2}$.
From the previous part, we have $D_{f_1} \cap R_{f_2} = [0, 2) \cap [4, \infty) = \emptyset$, so we cannot construct $f_1 \circ f_2$.

■

$\underline{f_2 \circ f_1}$ can be defined only when $D_{f_2} \cap R_{f_1}$.
From the previous part, we have $D_{f_2} \cap R_{f_1} = [2, \infty) \cap [0, 4) \neq \emptyset$, so we can construct $f_2 \circ f_1$: for $x \in D_{f_1}$,
$$(f_2 \circ f_1)(x) = f_2(f_1(x)) = f_2(x^2) = 3x^2 - 2.$$

For the domain $D_{f_2\circ f_1}$ of $f_2 \circ f_1$, we must have $D_{f_2\circ f_1} \subset D_{f_1}$ and also for $x \in D_{f_1}$, to have $f_2(f_1(x))$ well-defined, we must have $f_1(x) \in D_{f_2}$:

$$f_1(x) \geq 2 \qquad \Rightarrow \qquad x^2 \geq 2 \qquad \Rightarrow \qquad x \geq \sqrt{2} \text{ or } x \leq -\sqrt{2}.$$

So the domain $D_{f_2\circ f_1}$ is $\big\{(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)\big\} \cap D_{f_1} = \big\{(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)\big\} \cap [0, 2] = [\sqrt{2}, 2]$. From $(f_2 \circ f_1)(x) = 3x^2 - 2$, for $x \in [\sqrt{2}, 2]$,

$$\sqrt{2} \leq x \leq 2 \quad \Rightarrow \quad 3{\cdot}2-2 \leq 3x^2-2 \leq 3{\cdot}4-2 \quad \Rightarrow \quad 4 \leq 3x^2-2 \leq 10 \quad \Rightarrow \quad (f_2\circ f_1)(x) \in [4, 10]$$

and the range of $f_2 \circ f_1$ is $[4, 10]$. ■

$\underline{f_1 \circ F}$ can be defined only when $D_{f_1} \cap R_F$.
we have $D_{f_1} \cap R_F = [0, 2) \cap [0, \infty) \neq \emptyset$, so we can construct $f_1 \circ F$: for $x \in D_F$,

$$(f_1 \circ F)(x) = f_1(F(x))$$

and we have to choose the formula for $F(x)$ such that

$$F(x) \in D_{f_1} \qquad \Leftrightarrow \qquad 0 \leq F(x) < 2 \qquad \Leftrightarrow \qquad F(x) = x^2.$$

Note that if we use $F(x) = 3x - 2$, we will have $F(x) \geq 4$ because $x \geq 2$. Hence,

$$(f_1 \circ F)(x) = f_1(x^2) = x^4.$$

The domain can be obtained from $x \in D_F$,

$$F(x) \in D_{f_1} \quad \Leftrightarrow \quad 0 \leq x^2 < 2 \quad \Leftrightarrow \quad x^2-2 < 0 \quad \Leftrightarrow \quad (x-\sqrt{2})(x+\sqrt{2}) < 0 \quad \Leftrightarrow \quad -\sqrt{2} < x < \sqrt{2}$$

which implies that the domain $D_{f_1\circ F}$ is $[0, \infty] \cap (-\sqrt{2}, \sqrt{2}) = [0, \sqrt{2})$. From $(f_1\circ F)(x) = x^4$, for $x \in [0, \sqrt{2})$,

$$0 \leq x < \sqrt{2} \qquad \Rightarrow \qquad 0 \leq x^4 < 4$$

and the range of $f_1 \circ F$ is $R_{f_1\circ F} = [0, 4)$.

(c) Are f_1 and f_2 injective? Explain.
Yes, both f_1 and f_2 are injective. It can be shown that f_1 is injective as follows.
Let $x_1, x_2 \in D_{f_1} = [0, 2)$. We want to show that

$$f_1(x_1) = f_1(x_2) \quad \text{implies} \quad x_1 = x_2.$$

Suppose $f_1(x_1) = f_1(x_2)$. Then, for $x_1, x_2 \in [0, 2)$,

$$\begin{aligned} f_1(x_1) &= f_1(x_2) \\ x_1^2 &= x_2^2 \\ x_1 &= x_2. \end{aligned}$$

Note that we cannot have $x_1 = -x_2$ since $x_1, x_2 \geq 0$. Hence, f_1 is injective. ■

It can be shown that f_2 is injective as follows.
Let $x_1, x_2 \in D_{f_2} = [2, \infty)$. We want to show that

$$f_2(x_1) = f_2(x_2) \quad \text{implies} \quad x_1 = x_2.$$

Suppose $f_2(x_1) = f_2(x_2)$. Then, for $x_1, x_2 \geq 0$,

$$\begin{aligned} f_2(x_1) &= f_2(x_2) \\ 3x_1 - 2 &= 3x_2 - 2 \\ x_1 &= x_2, \end{aligned}$$

by adding 2 and dividing by 3 throughout the second equality. Hence, f_2 is injective. ■

(d) Is the function F bijective? If so, find the **inverse function** of F.

Yes, F is bijective. This can be verified by showing that F is one-to-one and onto.

F is one-to-one.

This can be shown as follows.

By definition, for $x_1, x_2 \in D_F = [0, \infty)$, F is one-to-one if

$$F(x_1) = F(x_2) \quad \text{implies} \quad x_1 = x_2.$$

Since there are two formulas for F depending on the value of x, we will consider 3 cases for the values of x_1 and x_2: (i) $x_1, x_2 \in [0, 2)$, (ii) $x_1, x_2 \in [2, \infty)$, (iii) $x_1 \in [0, 2)$ $x_2 \in [2, \infty)$.

(i) For $x_1, x_2 \in [0, 2)$, we will use $F(x) = x^2$ and, just like f_1, we have

$$F(x_1) = F(x_2) \quad \Rightarrow \quad x_1^2 = x_2^2 \quad \Rightarrow \quad x_1 = x_2.$$

(ii) For $x_1, x_2 \in [2, \infty)$, we will use $F(x) = 3x - 2$ and, just like f_2, we have

$$F(x_1) = F(x_2) \quad \Rightarrow \quad 3x_1 - 2 = 3x_2 - 2 \quad \Rightarrow \quad x_1 = x_2.$$

(iii) For $x_1 \in [0, 2)$ and $x_2 \in [2, \infty)$, we will use the contrapositive of the definition:

$$x_1 \neq x_2 \quad \Rightarrow \quad F(x_1) \neq F(x_2).$$

In this case, since x_1 and x_2 are from a disjoint intervals, then $x_1 \neq x_2$. We see that,

$$x_1 \in [0, 2) \quad \Rightarrow \quad F(x_1) = x_1^2 \in [0, 4)$$

as also shown in part (a) for range of F. and

$$x_2 \in [2, \infty) \quad \Rightarrow \quad F(x_2) = 3x_2 - 2 \in [4, \infty).$$

That is, $F(x_1) \neq F(x_2)$ because they are in two disjoint intervals $[0, 4)$ and $[4, \infty)$.

From cases (i)-(iii), we can conclude that F is injective.

F is onto.

Since F has two formulas and they give function values in two different disjoint intervals: $[0, 4)$ and $[4, \infty)$, as shown above and as also shown in (a), we will verify that F is onto using two cases.

Let $y \in [0, \infty)$, which is the co-domain of F.

- If $y \in [0,4)$, we can find $x \in D_F = [0,\infty)$ as follows.
  Set $y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$. So,

$$F(x) = F\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y.$$

  Since $0 \le y < 4$, then $0 \le \sqrt{y} < 2$, and $x = \sqrt{y} \in [0,2) \subset D_f$.
- If $y \in [4,\infty)$, we can find $x \in D_F = [0,\infty)$ as follows.
  Set $y = 3x - 2 \quad \Rightarrow \quad x = \frac{y+2}{3}$. So

$$F(x) = F\left(\frac{y+2}{3}\right) = 3\left(\frac{y+2}{3}\right) - 2 = y.$$

  Since $y \in [4,\infty)$ or $y \ge 4$, then $\frac{y+2}{3} \ge \frac{4+2}{3} = 2$, and $x = \frac{y+2}{3} \in [2,\infty) \subset D_F$.

The above two cases cover all possible values of y in the co-domain $[0,\infty)$ of F. That is, for any given $y \in [0,\infty)$, we can find $x \in D_F$ such that $F(x) = y$ and hence, F is onto.
Since F is one-to-one and onto, then it is bijective. ■
We now will find the inverse, F^{-1}, of F, which satisfies

$$F(x) = y \quad \Leftrightarrow \quad F^{-1}(y) = x.$$

When we showed that F is onto, we have

- For $y \in [0,4)$, there is $x = \sqrt{y} \in [0,2)$ such that

$$F\left(\sqrt{y}\right) = y.$$

- For $y \in [4,\infty)$, there is $x = \frac{y+2}{3} \in [2,\infty)$ such that

$$F\left(\frac{y+2}{3}\right) = y.$$

That is, we can defined $F^{-1} : [0,\infty) \to [0,\infty)$ as

$$F^{-1}(y) = \begin{cases} \sqrt{y}, & x \in [0,4) \\ \frac{y+2}{3}, & y \in [4,\infty). \end{cases}$$

■

9. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$

   (b) $f(x) = x^5, \quad g(x) = x^{1/5}.$

   Answer: First notice that all domains and ranges of f and g for both (a) and (b) are the set of real numbers $\mathbb{R}$ and therefore it is possible to define $f \circ g$ and $g \circ f$.

(a) $f(x) = \frac{x}{\sqrt{x^2+1}}$, $\quad g(x) = x^3+1$. The function $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f \circ g)(x) = f(g(x)) = \frac{x^3+1}{\sqrt{(x^3+1)^2+1}}$$

and function $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by

$$(g \circ f)(x) = g(f(x)) = \left(\frac{x}{\sqrt{x^2+1}}\right)^3 + 1.$$

So we have $(f \circ g)(2) = \frac{2^3+1}{\sqrt{(2^3+1)^2+1}} = \frac{9}{\sqrt{82}}$ and $(g \circ f)(2) = \left(\frac{2}{\sqrt{2^2+1}}\right)^3 + 1 = \frac{8+5\sqrt{5}}{5\sqrt{5}}$. Notice that since $(f \circ g)(2) \neq (g \circ f)(2)$, then $f \circ g \neq g \circ f$. ∎

(b) $f(x) = x^5$, $\quad g(x) = x^{1/5}$. The function $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f \circ g)(x) = f(g(x)) = \left(x^{1/5}\right)^5 = x.$$

and function $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by

$$(g \circ f)(x) = g(f(x)) = \left(x^5\right)^{1/5} = x.$$

Notice that since $(f \circ g)(x) = (g \circ f)(x)$ for all $x \in \mathbb{R}$, then $f \circ g = g \circ f$. So we have $(f \circ g)(2) = (g \circ f)(2) = 2$ ∎

10. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.

(a) Compute $f \circ f$ and determine its domain.

(b) Determine whether f is bijective. If so, find the inverse function f^{-1} and $f \circ f^{-1}$.

Answer:

(a) The composite function $f \circ f$ can be defined when $D_f \cap R_f \neq \emptyset$. The domain of f is $D_f = \mathbb{R} - \{1\}$. To find the range of f, we have to find all possible values of $y = f(x)$, which can be found by first set $y = f(x) = \frac{2x+1}{1-x}$ and then solve y in term of x:

$$y = \frac{2x+1}{1-x} \quad \Leftrightarrow \quad y - yx = 2x+1 \quad \Leftrightarrow \quad yx + 2x = y - 1 \quad \Leftrightarrow \quad x = \frac{y-1}{y+2}$$

which implies that y can be any real number except -2 and the range is $R_f = \mathbb{R} - \{-2\}$. Since $D_f \cap R_f = (\mathbb{R} - \{1\}) \cap (\mathbb{R} - \{-2\}) \neq \emptyset$, we can define $f \circ f$ as:

$$(f \circ f)(x) = f\left(\frac{2x+1}{1-x}\right) = \frac{2\frac{2x+1}{1-x}+1}{1-\frac{2x+1}{1-x}} = \frac{\frac{4x+2+1-x}{1-x}}{\frac{1-x-2x-1}{1-x}} = \frac{3x+3}{-3x} = -\frac{x+1}{x}.$$

The domain of $D_{f \circ f}$ can be obtained from $x \in D_f = \mathbb{R} - \{1\}$ and using the fact that: we can evaluate $f(f(x))$, only when

$$f(x) \in D_f \quad \Rightarrow \quad f(x) \neq 1 \quad \text{or} \quad \frac{2x+1}{1-x} \neq -1 \quad \Leftrightarrow \quad 1-x = 2x+1 \quad \Leftrightarrow \quad 3x \neq 0 \quad \Leftrightarrow \quad x \neq 0.$$

That is, the domain of $f \circ f$ is $D_f \cap (\mathbb{R} - \{0\}) = \mathbb{R} - \{0, 1\}$ ∎

(b) The function f is bijective.

<u>**f is one-to-one**</u>. Let $x_1, x_2 \in D_f = \mathbb{R} - \{1\}$. We want to show that

$$f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2.$$

Suppose $f(x_1) = f(x_2)$. Then, for $x_1, x_2 \in \mathbb{R} - \{1\}$,

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{2x_1+1}{1-x_1} &= \frac{2x_2+1}{1-x_2} \\ (2x_1+1)(1-x_2) &= (2x_2+1)(1-x_1) \\ 2x_1+1-2x_1x_2-x_2 &= 2x_2+1-2x_2x_1-x_1 \\ 2x_1-x_2 &= 2x_2-x_1 \\ 3x_1 &= 3x_2 \\ x_1 &= x_2. \end{aligned}$$

That is, f is one-to-one. ■

<u>**f is onto**</u>. For a given $y \in \mathbb{R} - \{-2\}$, which is the co-domain, we can show that f is onto by first setting $y = f(x)$:

$$y = \frac{2x+1}{1-x} \quad \Leftrightarrow \quad y - yx = 2x+1 \quad \Leftrightarrow \quad yx + 2x = y - 1 \quad \Leftrightarrow \quad x = \frac{y-1}{y+2}.$$

Since $y \in \mathbb{R} - \{2\}$, then $y \neq -2$ and x is well-defined real number in $\mathbb{R}$. To show that $f \in \mathbb{R} - \{1\}$, we will try to set $1 = x = \frac{y-1}{y+2}$ and we will see that

$$y - 1 = y + 2 \quad \Rightarrow \quad -1 = 2,$$

which is impossible and so $x = \frac{y-1}{y+2} \neq 1$. That is, given $y \in \mathbb{R} - \{-2\}$, we can find $x = \frac{y-1}{y+2} \in D_f = \mathbb{R} - \{1\}$ such that

$$f(x) = f\left(\frac{y-1}{y+2}\right) = \frac{2\left(\frac{y-1}{y+2}\right)+1}{1-\left(\frac{y-1}{y+2}\right)} = \frac{\left(\frac{2y-2+y+2}{y+2}\right)}{\left(\frac{y+2-y+1}{y+2}\right)} = \frac{\left(\frac{3y}{y+2}\right)}{\left(\frac{3}{y+2}\right)} = y \tag{3}$$

and, hence, f is onto. ■

To find the inverse, we can set $y = f(x) = \frac{2x+1}{1-x}$ as we did earlier and

$$y = \frac{2x+1}{1-x} \quad \Leftrightarrow \quad y - yx = 2x+1 \quad \Leftrightarrow \quad yx + 2x = y - 1 \quad \Leftrightarrow \quad x = \frac{y-1}{y+2},$$

which implies that the inverse function $f^{-1} : \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}$ is given by

$$f^{-1}(y) = \frac{y-1}{y+2}$$

or, equivalently, $f^{-1}(x) = \frac{x-1}{x+2}$ (by changing the notation of the variable from y to x). Note that we can use the definition by using (3): $f\left(\frac{y-1}{y+2}\right) = y \quad \Leftrightarrow \quad f^{-1}(y) = \frac{y-1}{y+2}$. ■

$\underline{f \circ f^{-1}}$
Note that $D_f = R_{f^{-1}} = \mathbb{R} - \{1\}$ and $R_f = D_{f^{-1}} = \mathbb{R} - \{-2\}$. So $D_f \cap R_{f^{-1}} = \mathbb{R} - \{1\} \neq \emptyset$ and we can define $f \circ f^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ by

$$(f \circ f^{-1})(x) = f\left(\frac{x-1}{x+2}\right) = \frac{2\left(\frac{x-1}{x+2}\right)+1}{1-\left(\frac{x-1}{x+2}\right)} = \frac{\left(\frac{2x-2+x+2}{x+2}\right)}{\left(\frac{x+2-x+1}{x+2}\right)} = \frac{\left(\frac{3x}{x+2}\right)}{\left(\frac{3}{x+2}\right)} = x.$$

That is, $\boxed{(f \circ f^{-1})(x) = x}$ is an identity matrix on $\mathbb{R} - \{1\}$. Note that the above process is similar to (3). ■

11. Consider the relation f defined from X to Y, $X, Y \subseteq \mathbb{R}$,

$$f = \left\{(x, y) \in X \times Y \,\middle|\, x^3 + yx^2 - y = 0\right\}.$$

Note: It might be useful to write y as a function of x, i.e. $y = f(x)$, before answering some questions.

(a) Find the domain of f.
(b) Find x-intercepts and y-intercepts (if any).
(c) Determine the symmetry of f.
(d) Find the horizontal and vertical asymptotes for f (if any).
(e) Show that $y = -x$ is a slant asymptote of f.
(f) Find the critical number of f. Determine the intervals on which f is increasing and decreasing. Determine the extrema (maximum and minimum) of f.
(g) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).
(h) Sketch the curve of f.

Answer: Notice that we can rearrange as $y = \frac{x^3}{1-x^2}$. So we can write this as

$$f(x) = \frac{x^3}{1-x^2}$$

(a) Since $1 - x^2$ is the denominator,so we must have $1 - x^2 \neq 0$. The domain of f (which consists of possible values of x) is $\mathbb{R} - \{1, -1\}$. ■

(b) x-intercepts: setting $y = 0$ gives $0 = \frac{x^3}{1-x^2}$, which occurs if and only if

$$x^3 = 0 \quad \Leftrightarrow \quad x = 0.$$

So x-intercepts is $(0, 0)$.
To find y-intercepts, set $x = 0$, which implies $y = 0$. So $(0, 0)$ is also a y-intercept of f. ■

(c) Determine the symmetry of f.
Notice that f is a function and can be written as $f(x) = \frac{x^3}{1-x^2}$. Hence, it is not symmetric about the x-axis. We will look at the symmetry by using $f(-x)$.

$$f(-x) = \frac{(-x)^3}{1-(-x)^2} = -\frac{x^3}{1-x^2}.$$

Since $f(-x) = -f(x)$, f is an odd function and it is symmetric about the origin.
Note that $f(-x) \neq f(x)$, it is not symmetric about the y-axis. ■

(d) Find the horizontal and vertical asymptotes for f (if any).
Horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3}{1-x^2} = \infty$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{1-x^2} = -\infty$$

So, there is no horizontal asymptote. ■

Vertical asymptotes: Notice that the function is not defined at $x = \pm 1$.

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{x^3}{1-x^2} = \infty \qquad \lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{x^3}{1-x^2} = -\infty$$

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} \frac{x^3}{1-x^2} = -\infty \qquad \lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \frac{x^3}{1-x^2} = \infty$$

So, the vertical asymptotes are $x = -1$ and $x = 1$. ■

(e) To show that $y = -x$ is a slant asymptote, we must have $\lim_{x\to\infty}(f(x) - (-x)) = 0$:

$$\lim_{x\to\infty}(f(x) - (-x)) = \lim_{x\to\infty} \frac{x^3}{1-x^2} + x = \lim_{x\to\infty} \frac{x^3 + x - x^3}{1-x^2} = \lim_{x\to\infty} \frac{x}{1-x^2} = 0$$

That is, $y = -x$ is a slant asymptote by the limit definition. ■

Alternative answer: Suppose the slant asymptote is in the form of $y = mx + b$. We will determine this asymptote by finding the values of m and b such that

$$\lim_{x\to\infty}(f(x) - [mx + b]) = 0.$$

From $f(x) = \frac{x^3}{1-x^2}$,

$$\begin{aligned}
\lim_{x\to\infty}(f(x) - [mx+b]) &= \lim_{x\to\infty}\left(\frac{x^3}{1-x^2} - [mx+b]\right) = \lim_{x\to\infty} \frac{x^3 - (mx+b)(1-x^2)}{1-x^2} \\
&= \lim_{x\to\infty} \frac{x^3 - (mx - mx^3) - (b - bx^2)}{1-x^2} \\
&= \lim_{x\to\infty} \frac{(1+m)x^3 - bx^2 - mx - b}{1-x^2},
\end{aligned}$$

which will become zero only when the degree of x in the numerator is **smaller** than the degree of x in the denominator. Hence the coefficients of x^3 and x^2 in the numerator have to be zero:

$$1 + m = 0 \qquad \text{and} \qquad b = 0$$

or $\boxed{m = -1 \text{ and } b = 0}$. By substituting $m = -1, b = 0$ in $y = mx + b$, we have that $y = -x$ is a slant asymptote of the curve $y = f(x)$. ■

(f) Find the critical number of f.

$$f'(x) = \frac{3(1-x^2)x^2 - (-2x)x^3}{(1-x^2)^2} = \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2} = \frac{3x^2 - x^4}{(1-x^2)^2} = \frac{x^2(3-x^2)}{(1-x^2)^2}$$

The critical number x occurs when $f'(x) = 0$ or $f'(x)$ does not exist.

$$f'(x) = 0 \quad \Leftrightarrow \quad 3 - x^2 = 0, x^2 = 0 \quad \Leftrightarrow \quad x = -\sqrt{3}, 0, \sqrt{3}.$$

Note that $f'(x)$ does not exist when $x = \pm 1$, which is not in the domain, so ± 1 is not a critical number. Hence, the critical numbers are $x = -\sqrt{3}, 0, \sqrt{3}$. ■

♦ Determine the intervals on which f is increasing and decreasing. Determine the relative extrema (maximum and minimum) of f.
From

$$f'(x) = \frac{x^2(3-x^2)}{(1-x^2)^2} = \frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2}$$

- Decreasing: $f'(x) < 0$ when $\frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2} < 0$
- Increasing: $f'(x) > 0$ when $\frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2} > 0$

We will consider 6 intervals from $x = -\sqrt{3}, -1, 0, 1, \sqrt{3}$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
Sign of $f'(x) = \frac{x^2(\sqrt{3}+x)(\sqrt{3}-x)}{(1-x^2)^2}$ $-(x+\sqrt{3})(x-\sqrt{3})$	$-$ $\frac{(+)(-)(+)}{(+)}$	$+$ $\frac{(+)(+)(+)}{(+)}$	$+$ $\frac{(+)(+)(+)}{(+)}$	$+$ $\frac{(+)(+)(+)}{(+)}$	$+$ $\frac{(+)(+)(+)}{(+)}$	$-$ $\frac{(+)(+)(-)}{(+)}$

I.e., $f'(x) < 0$ when $x < -\sqrt{3}$ and $x > \sqrt{3}$. Also, $f'(x) > 0$ for $x \in (-\sqrt{3}, \sqrt{3})$ or $x \in (-\sqrt{3}, -1) \cup (-1, 1) \cup (1, \sqrt{3})$(note: $x = \pm 1$ are not in the domain)

- Decreasing interval: $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$
- Increasing interval: $(-\sqrt{3}, -1) \cup (-1, 1) \cup (1, \sqrt{3})$.

■

Therefore, from the *First Derivative Test*,
- relative maximum occurs at $x = \sqrt{3}$ (the sign of $f'(x)$ changes from "+" to "−" at $x = \sqrt{3}$)
and $f(\sqrt{3}) = -\frac{3\sqrt{3}}{2}$ is the relative maximum
- relative minimum occurs at $x = -\sqrt{3}$ (the sign of $f'(x)$ changes from "−" to "+" at $x = -\sqrt{3}$) and
$f(-\sqrt{3}) = \frac{3\sqrt{3}}{2}$ is the relative minimum.

■

(g) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).
From $f'(x) = \frac{3x^2-x^4}{(1-x^2)^2}$,

$$\begin{aligned} f''(x) &= \frac{(1-x^2)^2(6x-4x^3)-2(1-x^2)(-2x)(3x^2-x^4)}{(1-x^2)^4} \\ &= 2x\frac{(1-x^2)(3-2x^2)+6x^2-2x^4}{(1-x^2)^3} \\ &= \frac{2x(3+x^2)}{(1-x^2)^3} \end{aligned}$$

So $f''(0) = 0$ when $x = 0$. We will check if $x = 0$ is a inflection point, we have to see if the the sign of $f''(x)$ is changed at $x = 0$ or not.
To determine the curvature, we have to solve the inequalities $f''(x) < 0$ and $f''(x) > 0$, which can be done by considering the zeros of the numerator $2x(3+x^2)$ and $(1-x^2)^3$ or at $x = 0, \pm 1$. That is, we consider the intervals $(-\infty,-1)$,$(-1,0)$,$(0,1)$,$(1,\infty)$.

Interval of x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Sign of $f''(x) = \frac{2x(3+x^2)}{(1-x^2)^3}$	$+$	$-$	$+$	$-$
$\frac{2x(3+x^2)}{(1-x^2)^3}$	$\frac{(-)(+)}{(-)}$	$\frac{(-)(+)}{(+)}$	$\frac{(+)(+)}{(+)}$	$\frac{(+)(+)}{(-)}$

- Concave up: $f'' > 0$ when $x \in (-\infty,-1) \cup (0,1)$.
- Concave down: $f'' < 0$ when $x \in (-1,0) \cup (1,\infty)$.

■

From above, we have that the sign of $f''(x)$ changes at $x = 0$, so the point $(0,0)$ is the inflection point. Note that $x = -1, 1$ don't give inflection points because they are not in the domain (even though the sign of $f''(x)$ changes at these points). ■

(h) Sketch the curve of f.

12. Find all solution(s) of the following systems of linear equations by using **Gaussian elimination method**.

(a)
$$\begin{array}{rcrcrcrcr} x_1 &+& x_2 &+& x_3 &+& x_4 &=& 6 \\ x_1 &+& 2x_2 &+& 3x_3 &+& 4x_4 &=& 16 \\ 2x_1 &+& 3x_2 &+& 5x_3 &+& 6x_4 &=& 25 \\ x_1 &+& x_2 &+& 2x_3 &+& 3x_4 &=& 11 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 1 \\ 3x &+& 2y &+& z &=& 1 \\ 7x &+& 2y &-& 3z &=& 1 \end{array}$$

<u>**Answer:**</u>

(a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

The augmented form is given by $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 & 16 \\ 2 & 3 & 5 & 6 & 25 \\ 1 & 1 & 2 & 3 & 11 \end{array}\right]$.

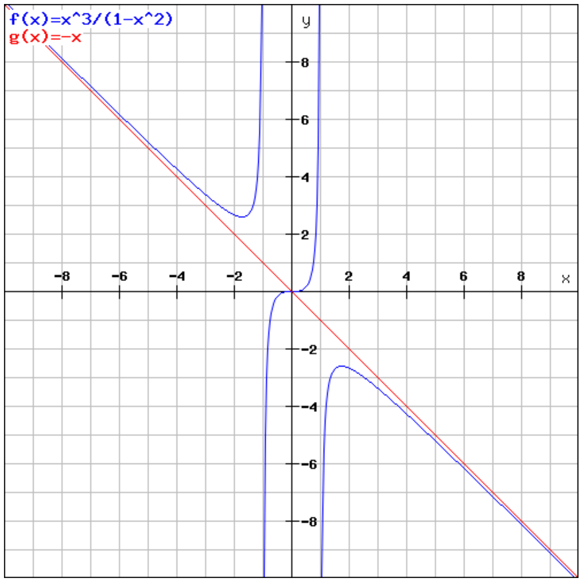

Figure 3: $f = \{(x, y)|x^3 + yx^2 - y = 0\}$

Step 1: R_1 is the "*pivot row*" and $a_{11} = 1$ is the "*pivot element*".
Let $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{1} = 1$, $m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$, and $m_{41} = \frac{a_{41}}{a_{11}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 \mapsto R_2 - m_{21}R_1 : \\ R_3 \mapsto R_3 - m_{31}R_1 : \\ R_4 \mapsto R_4 - m_{41}R_1 : \end{array} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 3 & 10 \\ 0 & 1 & 3 & 4 & 13 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

Step 2: R_2 is the "*pivot row*" and $a_{22} = 1$ is the "*pivot element*".
Let $m_{32} = \frac{a_{32}}{a_{22}} = \frac{1}{1} = 1$ and $m_{42} = \frac{a_{42}}{a_{22}} = \frac{0}{1} = 0$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 \mapsto R_3 - m_{32}R_2 : \\ R_4 \mapsto R_4 - m_{42}R_2 : \end{array} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 3 & 10 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

Step 3: R_3 is the "*pivot row*" and $a_{33} = 1$ is the "*pivot element*".
Let $m_{43} = \frac{a_{43}}{a_{33}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \\ R_4 \mapsto R_4 - m_{43}R_3 : \end{array} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 3 & 10 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

This gives the following equivalent system.

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 6 \\ & & x_2 & + & 2x_3 & + & 3x_4 & = & 10 \\ & & & & x_3 & + & x_4 & = & 3 \\ & & & & & & x_4 & = & 2 \end{array}$$

By applying back substitution, we have

R_4: $x_4 = 2$

R_3: $x_3 + 2 = 3 \implies x_3 = 1$

R_2: $x_2 + 2(1) + 3(2) = 10 \implies x_2 = 2$

R_1: $x_1 + 2 + 1 + 2 = 6 \implies x_1 = 1$

■

(b) From $\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 3x & + & 2y & + & z & = & 1 \\ 7x & + & 2y & - & 3z & = & 1 \end{array}$, the augmented form is $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \\ 7 & 2 & -3 & 1 \end{array}\right]$.

Step 1: R_1 is the "*pivot row*" and $a_{11} = 1$ is the "*pivot element*".
Let $m_{21} = \frac{a_{21}}{a_{11}} = \frac{3}{1} = 3$ and $m_{31} = \frac{a_{31}}{a_{11}} = \frac{7}{1} = 7$.

$$\begin{array}{l} R_1 : \\ R_2 \mapsto R_2 - m_{21}R_1 : \\ R_3 \mapsto R_3 - m_{31}R_1 : \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \\ 0 & -12 & -24 & -6 \end{array}\right]$$

Step 2: R_2 is the "*pivot row*" and $a_{22} = -4$ is the "*pivot element*".
Let $m_{32} = \frac{a_{32}}{a_{22}} = \frac{-12}{-4} = 3$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 \mapsto R_3 - m_{32}R_2 : \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By applying back substitution, we have

R_3: z can be any real number, say, $z = t, t \in \mathbb{R}$.

R_2: $-4y - 8z = -2 \implies y = \frac{1}{2} - 2t,\ t \in \mathbb{R}$

R_1: $x + 2y + 3z = 1 \implies x = 1 - 2(\frac{1}{2} - 2t) - 3t = t$.

Hence there are infinitely many solutions and can be given by $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ \frac{1}{2} - 2t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$.

■

13. Consider the system of 3 equations and 3 unknowns x, y, z:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ x & + & 3y & + & 4z & = & 3 \\ x & + & 4y & + & kz & = & m, \end{array}$$

where m and k are some constants.

(a) Suppose $k = 5$. Find the value of m that makes the above system of equations have infinitely many solutions.

(b) Suppose $m = 4$. Find the value of m that makes the above system of equations have no solution.

Answer: By performing Gaussian elimination method, we have:

(a) $m = 5$.

(b) $k = 5$.

14. Write out the appropriate form of the partial fraction decomposition of the given expressions Do not evaluate the coefficients.

(a)

$$\frac{x+1}{2x^2 - 11x + 15}$$

Answer:

$$\frac{x+1}{2x^2 - 11x + 15} = \frac{x+1}{(x-3)(2x-5)} = \frac{A}{x-3} + \frac{B}{2x-5},$$

where A, B are constants.

(b)

$$\frac{1}{x^4(x-1)(x^2+4)^3(x^2+x+1)^2}$$

Answer: Note that, $x^2 + 4$ and $x^2 + x + 1$ are irreducible polynomial (this can be checked by using $b^2 - 4ac < 0$ for $ax^2 + bx + c$). So the partial fraction decomposition of $\frac{1}{x^4(x-1)(x^2+4)^3(x^2+x+1)^2}$ is in the form

$$= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4}{x^4} + \frac{B}{x-1} + \frac{C_1 x + D_1}{x^2+4} + \frac{C_2 x + D_2}{(x^2+4)^2} + \frac{C_3 x + D_3}{(x^2+4)^3} + \frac{E_1 x + F_1}{x^2+x+1} + \frac{E_2 x + F_2}{(x^2+x+1)^2},$$

where $A_1, A_2, A_3, A_4, B, C_1, C_2, C_3, D_1, D_2, D_3, E_1, E_2, F_1, F_2$ are constants.

15. Determine the partial fraction decomposition of

$$\frac{x^4 + 7x^3 + 8x^2 - 3}{x^3 + 6x^2}.$$

Answer: Note that the given expression is not in the form of proper fraction. Using the long division, we have

$$\frac{x^4 + 7x^3 + 8x^2 - 3}{x^3 + 6x^2} = x + 1 + \frac{2x^2 - 3}{x^2(x+6)}$$

So we will apply partial fraction decomposition on $\frac{2x^2-3}{x^2(x+6)}$ which will be in the form:

$$\frac{2x^2 - 3}{x^2(x+6)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+6}.$$

We can solve for A, B, C as follows.

$$\begin{aligned}
\frac{2x^2-3}{x^2(x+6)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+6} = \frac{Ax(x+6)+B(x+6)+Cx^2}{x^2(x+6)} \\
2x^2-3 &= Ax(x+6)+B(x+6)+Cx^2 \\
&= \underbrace{[A+C]}_{=2}x^2 + \underbrace{[6A+B]}_{=0}x + \underbrace{6B}_{=-3},
\end{aligned}$$

which gives 3 equations

$$A+C=2 \qquad 6A+B=0 \qquad 6B=-3$$

and so $A = 1/12, B = -1/2, C = 23/12$. So, the partial fraction decomposition is

$$\frac{x^4+7x^3+8x^2-3}{x^3+6x^2} = x+1+\frac{1}{12x} - \frac{1}{2x^2} + \left(\frac{23}{12}\right)\frac{1}{x+6}.$$

■