Solution: Exercise 3 (Part 1)

1. (a) Prove the statement:
"There is a pair of real numbers x and y such that $\lfloor x-y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. "

(b) Disprove the statement: "For all real numbers x and y, $\lfloor x-y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$. "

Answer:

(a) To prove this *existential statement,* we can find a pair of x and y such that the given statement is true.
Let $x = 0$ and $y = 0$. Then $\lfloor x-y \rfloor = \lfloor 0-0 \rfloor = \lfloor 0 \rfloor = 0$, and $\lfloor x \rfloor = \lfloor 0 \rfloor = 0$ and $\lfloor y \rfloor = \lfloor 0 \rfloor = 0$. That is,

$$\lfloor 0-0 \rfloor = \lfloor 0 \rfloor - \lfloor 0 \rfloor$$

and hence the statement is true. ■

(b) To disprove the *universal statement*, we can find a pair of x and y such that the given statement is false.
Consider

$$\lfloor 1 \rfloor = 1, \quad \lfloor 0.5 \rfloor = 0, \text{ which imply } \lfloor 1 \rfloor - \lfloor 0 \rfloor = 1$$

but

$$\lfloor 1-0.5 \rfloor = \lfloor 0.5 \rfloor = 0.$$

That is, a counterexample is $x = 1, y = 0.5$. ■

2. Show that "for any integer n, if n^3+5 is odd, then n is even," by using
a) a proof by contraposition,
b) a proof by contradiction.

Answer

a) **Proof by contraposition**: To prove by a contraposition, we consider the contrapositive of the given statement:

for any integer n, if n is odd, then n^3+5 is even.

We can do this by direct proof. Suppose n is odd. Then, we can write

$$n = 2k+1,$$

where k is an integer and

$$n^3+5 = (2k+1)^3+5 = (2k)^3+3(2k)^2+3(2k)+1+5 = 2(4k^3+6k^2+3k+3).$$

That is, we can write n^3+5 in terms of $n^3+5 = 2s$ where $s = 4k^3+6k^2+3k+3$ is an integer (since it is the product and the sum of integers). Hence, n^3+5 is an even integer. ■

b) **Proof by contradiction**:

Suppose not. I.e., suppose the negation " $n^3 + 5$ is odd, but n is not even" is true. Then, n is odd and we can write

$$n = 2k + 1$$

for some integer $k \in \mathbb{Z}$. So we have

$$n^3 + 5 = (2k+1)^3 + 5 = (2k)^3 + 3(2k)^2 + 3(2k) + 1 + 5 = 2(4k^3 + 6k^2 + 3k + 3),$$

which implies that $n^3 + 5$ is an even integer (since we can write $n^3 + 5$ in terms of $n^3 + 5 = 2s$ where $s = 4k^3 + 6k^2 + 3k + 3$ is an integer).
This **contradicts** to the fact that n^3+5 is odd. Hence, the negation is false and cannot happen. That is, the given statement is true. ■

3. Prove by the **method of exhaustion** that " $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$."

   Answer Given that an integer n such that $1 \leq n \leq 4$ implies that $n = 1, 2, 3,$ or 4.

   For $n = 1$, $1^2 + 1 = 2^1$ and $n^2 + 1 \geq 2^n$ is true.
   For $n = 2$, $2^2 + 1 = 5$ and $2^2 = 4$, so $n^2 + 1 \geq 2^n$ is true.
   For $n = 3$, $3^2 + 1 = 10$ and $2^3 = 8$, so $n^2 + 1 \geq 2^n$ is true.
   For $n = 4$, $4^2 + 1 = 17$ and $2^4 = 16$, so $n^2 + 1 \geq 2^n$ is true.

   Therefore, $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$. ■

4. Use the **proof by cases** to show that " for any integer n, $n^2 \geq n$."
   [Hint: Consider 3 cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$]

   Answer We will prove by cases: (i) $n \in \mathbb{Z}^-$, (ii) $n = 0$, (iii) $n \in \mathbb{Z}^+$.

   Case (i): $n \in \mathbb{Z}^-$
   In this case, $n < 0$ and $n^2 > 0$. That is $n < 0 < n^2$ and $n^2 \geq n$ is true.

   Case (ii): $n = 0$
   In this case, $n^2 = 0$. So $n = n^2$ and $n^2 \geq n$ is true.

   Case (iii): $n \in \mathbb{Z}^+$
   In this case, consider $n^2 - n = n(n-1)$. Since $n \in \mathbb{Z}^+$ implies that n is a positive integer and the smallest value of n is 1. That is $n \geq 1$, which implies $n - 1 \geq 0$. Since $n > 0$ and $n - 1 \geq 0$,

$$n^2 - n = n(n-1) \geq 0, \text{ which imples} n^2 \geq n,$$

   and the given statement is true.

   Since any integer n can be in either case (i), (ii), or (iii), then the given statement is true. ■

5. Consider the statement: for $n \geq 1$,

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

   Suppose we want to prove the above statement by **mathematical induction**.

(a) What is $P(n)$?

(b) Write $P(1)$: Is $P(1)$ true?

(c) Write $P(k)$:

(d) Write $P(k+1)$:

(e) Prove the above statement: $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$, by using **mathematical induction**.

Answer:

(a) $P(n)$ is a statement

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n},$$

for $n \geq 1$

(b) Write $P(1)$:

$$\frac{1}{2} = \frac{2^(1-1)}{2^1},$$

Yes, $P(1)$ is true because $\frac{2^(1-1)}{2^1} = \frac{1}{2}$.

(c) $P(k)$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

(d) $P(k+1)$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}.$$

(e) Prove the above statement: $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$, by using **mathematical induction**.
Let $P(n)$ be the statement $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$.

(I) Basis step: Show that $P(1)$ is true. $P(1)$:

$$\frac{1}{2} = \frac{2^(1-1)}{2^1},$$

$P(1)$ is true because $\frac{2^(1-1)}{2^1} = \frac{1}{2}$.

(II) Inductive step: Show that if $P(k)$ is true, then $P(k+1)$ is true.
Assume that $P(k)$ is true, or

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

We want to show that $P(k+1)$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}.$$

$$
\begin{aligned}
\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{k+1}} &= \underbrace{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^k}}_{=\frac{2^k-1}{2^k}}+\cdots+\frac{1}{2^{k+1}} \quad \text{by inductive hypothesis } P(k)\\
&= \frac{2^k-1}{2^k}+\frac{1}{2^{k+1}}\\
&= \frac{2(2^k-1)+1}{2^{k+1}}\\
&= \frac{(2^{k+1}-2)+1}{2^{k+1}}\\
&= \frac{2^{k+1}-1}{2^{k+1}}
\end{aligned}
$$

and the statement $P(k+1)$ is true.
From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 1$ by mathematical induction proof. ■

6. Use mathematical induction proof to show that

$$n! < n^n,$$

for any integer n that is greater than 1.

Answer:

Proof by mathematical induction:
Let $P(n)$ be the statement $n! < n^n$. We want to prove that $P(n)$ is true for all integer $n > 1$. Note that for $n \in \mathbb{Z}$, $n > 1$ is equivalent to $n \geq 2$ and we therefore have to use $n = 2$ in the basis step.

(I) **Basis step:** Show that $P(2)$ is true.
$P(2)$: $2! < 2^2$.
Since $2! = 2$ and $2^2 = 4$. Hence $2! < 2^2$ and $P(2)$ is true.

(II)**Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 2$.

Assume that $P(k) : k! < k^k$ is true. ————————(★) "inductive hypothesis"
We want to show that $P(k+1) : (k+1)! < (k+1)^{(k+1)}$ is true. Consider

$$
\begin{aligned}
(k+1)! &= k!(k+1)\\
&< k^k(k+1) && \text{by (★) "inductive hypothesis"}\\
&< (k+1)^k \cdot (k+1) && k^k < (k+1)^k \text{ for } k \geq 2\\
&= (k+1)^{(k+1)}
\end{aligned}
$$

Note: we have used the fact that since $k < k+1$ implies $k^k < (k+1)^k$ (using the same exponent). Therefore $P(k+1)$ is true.
Note also that we have used $k!(k+1) = \underbrace{1 \cdot 2 \cdot 3 \cdots k}_{k!} \cdot (k+1) = (k+1)!$.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 2$ by mathematical induction proof. ■

7. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

8. (Optional) Use the method of constructive proof to show that:
   if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

   Answer: Constructive proof
   Let $r, s \in \mathbb{R}$ such that $r < s$. Let

   $$x = \frac{r+s}{2}.$$

   We will show that for this particular x has the value between the r and s.

   $$\begin{aligned} r &< s \\ r + r &< s + r \\ \underbrace{\frac{r+r}{2}}_{=r} &< \underbrace{\frac{s+r}{2}}_{=x} \quad \Rightarrow \quad r < x \end{aligned}$$

   $$\begin{aligned} r &< s \\ r + s &< s + s \\ \underbrace{\frac{r+s}{2}}_{=x} &< \underbrace{\frac{s+s}{2}}_{=s} \quad \Rightarrow \quad x < s \end{aligned}$$

   That is, for any given r and s, we can always find $x = \frac{r+s}{2}$ such that $r < x < s$.

   Note that it is also possible to use a different value of x. ■

9. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

   Answer: Let r be any rational number and s be any irrational number. We want to show that $r - s$ is irrational.

   To prove this by contradiction, we will suppose that $r - s$ is rational. Then we can write $r = \frac{a}{b}$ and $r - s = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$. That is,

   $$\frac{a}{b} - s = \frac{c}{d}$$

   and so

   $$s = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

   which implies that s is rational. This is a contradiction to the assumption that s is irrational. Therefore, the the given statement is true by contradiction proof. ■

10. (Optional) A sequence a_1, a_2, ... is defined recursively by

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$

Show that

$$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$

Answer:

Proof by mathematical induction:

From the given definition :

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2. \qquad - - - - - - - - - - - - - - - \circledast$$

Let $P(n)$ be the statement $a_n = 3 \cdot 7^{n-1}$.
We want to prove that $P(n)$ is true for all integer $n \geq 1$.

(I) **Basis step:** Show that $P(1)$ is true.
$P(1)$: $a_1 = 3 \cdot 7^{1-1}$.
Since $a_1 = 3 \cdot 7^{1-1} = 3 \cdot 7^0 = 3 \cdot 1 = 3$, which is the same as $a_1 = 3$ from the definition $\circledast$.
Hence $P(1)$ is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.

Assume that $P(k) : a_k = 3 \cdot 7^{k-1}$ is true. ————————($\bigstar$) "inductive hypothesis"
We want to show that $P(k+1) : a_{k+1} = 3 \cdot 7^{(k+1)-1}$, or $a_{k+1} = 3 \cdot 7^k$ is true. Consider from the definition $\circledast$

$$\begin{aligned} a_{k+1} &= 7a_{[(k+1)-1]} \\ &= 7\ a_k \\ &= 7\ [3 \cdot 7^{k-1}] \qquad \text{by } (\bigstar) \text{ "inductive hypothesis : } a_k = 3 \cdot 7^{k-1} \text{"} \\ &= 3 \cdot 7^k \end{aligned}$$

and therefore $P(k+1)$ is true.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 1$ by mathematical induction proof. ■