1. *Theory of firm*

Suppose that production function is given by $Q = f(K, L) = \alpha\sqrt{K} + \beta L$ where K and L are the unit of capital installed and the number of employees hired, respectively. Assume that price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

a) The firm wants to minimize cost and seek for combination of the two factor inputs to produce output level Q_0. Derive the factor inputs demand.

min cost : $wL + rK$

s.t. production function $\bar{Q} = f(K, L) = \alpha\sqrt{K} + \beta L$

$\mathcal{L} = wL + rK - \lambda(\bar{Q} - \alpha\sqrt{K} + \beta L)$

foc ; $\mathcal{L}_K = \frac{\partial \mathcal{L}}{\partial K} = r - \lambda \frac{1}{2} K^{-\frac{1}{2}} = 0$ — (1)

$\mathcal{L}_L = w - \lambda\beta = 0$ — (2) $\rightarrow \lambda^* = \frac{w}{\beta}$

$\mathcal{L}_\lambda = \bar{Q} - \alpha\sqrt{K} - \beta L$ — (3)

$\frac{(1)}{(2)}$; $\frac{r}{w} = \frac{\lambda \frac{1}{2} \alpha K^{-\frac{1}{2}}}{\lambda\beta} = \frac{\alpha}{2\beta\sqrt{K}}$ — (4)

find K^* ; $\left(\frac{r}{w}\right)^2 = \left(\frac{\alpha}{2\beta\sqrt{K}}\right)^2$

$\left(\frac{r}{w}\right)^2 = \frac{\alpha^2}{4\beta K}$

$= \frac{\alpha^2}{4\beta^2\left(\frac{r}{w}\right)^2}$

$K^* = \left(\frac{w\alpha}{2\beta r}\right)^2$

find L^* ; sub K^* into $\bar{Q} - \alpha\sqrt{K} - \beta L$

$\bar{Q} - \alpha\sqrt{\left(\frac{w\alpha}{2\beta r}\right)^2} - \beta L = 0$

$\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right) - \beta L = 0$

$L^* = \frac{\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right)}{\beta}$

b) Confirm your result with the second order derivative test.

check SOC ;

hessian matrix

$$H \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda K} & L_{\lambda L} \\ L_{K\lambda} & L_{KK} & L_{KL} \\ L_{L\lambda} & L_{LK} & L_{LL} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha\frac{1}{2}k^{-\frac{1}{2}} & -\beta \\ -\alpha\frac{1}{2}k^{\frac{1}{2}} & \frac{1}{4}\lambda\alpha k^{-\frac{3}{2}} & 0 \\ -\beta & 0 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|cc} 0 & -\alpha\frac{1}{2}k^{-\frac{1}{2}} & -\beta & 0 & -\alpha\frac{1}{2}k^{\frac{1}{2}} \\ -\alpha\frac{1}{2}k^{\frac{1}{2}} & \frac{1}{4}\lambda\alpha k^{-\frac{3}{2}} & 0 & -\alpha\frac{1}{2}k^{\frac{1}{2}} & \frac{1}{4}\lambda\alpha k^{-\frac{3}{2}} \\ -\beta & 0 & 0 & -\beta & 0 \end{array}\right]$$

$$= (0+0+0) - \left[\beta^2 \cdot \left(\frac{1}{4}\lambda\alpha k^{-\frac{3}{2}}\right)\right]$$

$$= -\left[\beta^2 \cdot \left(\frac{1}{4}\lambda\alpha k^{-\frac{3}{2}}\right)\right] < 0$$

- convex function
- confirmed minimized cost-func.

c) State the condition under which demand for capital and labor are both strictly positive.

$$k^* = \left(\frac{W\alpha}{2\beta r}\right)^2$$

must be more than zero, and for the condition to exist β and r must not equal zero.

$$L^* = \frac{\bar{Q} - \frac{W\alpha^2}{2\beta r}}{\beta}$$

→ This value must be more than zero

for the condition to exist β and r must not equal zero.

d) Suppose that the condition required for strictly positive solution holds, derive the long-run optimal cost function, and show that marginal cost function is equal to the LaGrange multiplier of your cost minimization problem.

$$K^* = \left(\frac{w\alpha}{2\beta r}\right)^2 \qquad L^* = \frac{\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right)}{\beta}$$

$$C = w \cdot L + r \cdot K$$

$$C^* = C(K^*, L^*)$$

$$C^* = w\left[\frac{\bar{Q} - \alpha\left(\frac{w\alpha}{2\beta r}\right)}{\beta}\right] + r \cdot \left(\frac{w\alpha}{2\beta r}\right)^2$$

prove $\lambda^* = \frac{\partial C^*}{\partial \bar{Q}}$

$(K) \Rightarrow \mathcal{L}_K = r - \lambda \frac{1}{2}\alpha K^{-\frac{1}{2}} = 0$ — (1)

$(L) \Rightarrow \mathcal{L}_L = w - \lambda B = 0$ — (2) $\Rightarrow \lambda^* = \frac{w}{\beta}$

$(\lambda) \Rightarrow \mathcal{L}_\lambda = \bar{Q} - \alpha\sqrt{K} - \beta L$ — (3)

marginal cost: $\frac{\partial C^*}{\partial \bar{Q}} = \frac{w}{\beta}$ and $\lambda^* = \frac{w}{\beta}$ #

2. *Theory of consumer*

Consider a household with the utility function given by,

$$U(x, y) = [x^2 + y^2]^{\frac{1}{2}}$$

where x *and* y are two different consumption goods, i.e. good x and good y. Suppose that (i) *the prices for each of the two consumption goods are* p_x *and* p_y *respectively*, and (ii) *household's income is equal to M.* Consider the following problems

a) Calculate the total differential of the utility function.

b) Set up the constrained optimization problem and derive the Marshallian demand function.

c) Does the demand function satisfy *the law of demand*? Mathematically, how do you know that?

d) How does the demand for good y respond to price of good x?

e) What is the numerical value of λ^* when $M = \$300,\ p_x = 1,\ p_y = 1$?

f) Without redoing the optimization problem, what would be the new optimized level of maximum utility when income increases to \$310.

a) $du(x,y) = \frac{\partial u(x,y)}{\partial x} dx + \frac{\partial u(x,y)}{\partial x} dy$

$$= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x \cdot dx + \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y \cdot dy$$

$$= (x^2+y^2)^{-1/2} dx + (x^2+y^2)^{-1/2} dy \quad \#$$

b) Marshallian demand function : max $U(x,y) = (x^2+y^2)^{1/2}$

$\mathcal{L} = (x^2+y^2)^{1/2} + \lambda(m - P_x \cdot x + P_y \cdot y)$ s.t. $P_x x + P_y \cdot y = m$

FOC : $\frac{\partial \mathcal{L}}{\partial x} = 0$; $\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x + \lambda(-P_x) = 0$ —— (1)

$\frac{\partial \mathcal{L}}{\partial y} = 0$; $\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y + \lambda(-P_y) = 0$ —— (2)

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$; $m - P_x \cdot x - P_y \cdot y = 0$ —— (3)

$(x^2+y^2)^{-1/2} \cdot x = \lambda P_x$ —— (4)

$(x^2+y^2)^{-1/2} \cdot y = \lambda P_y$ —— (5)

$\frac{(4)}{(5)}$; $\frac{(x^2+y^2)^{-1/2} \cdot x}{(x^2+y^2)^{-1/2} \cdot y} = \frac{\lambda P_x}{\lambda P_y} = \frac{x}{y} = \frac{P_x}{P_y}$

$y = x \cdot \frac{P_y}{P_x}$ —— (6)

sub (6) into (3) ; $m - P_x \cdot x - P_y \cdot x \cdot \frac{P_y}{P_x} = 0$

$$M = P_x \cdot x + P_y^2 \cdot \frac{x}{P_x}$$

$$M = x\left(P_x + \frac{P_y^2}{P_x}\right)$$

$$\frac{M}{\left(P_x + \frac{P_y^2}{P_x}\right)} = x$$

$$x^* = \frac{M \cdot P_x}{P_x^2 + P_y^2}$$

sub $x = \frac{P_x}{P_y} \cdot y$ into (3) ; $M - P_x \cdot \left(\frac{P_x}{P_y} \cdot y\right) - P_y \cdot y = 0$

$$M = \frac{P_x^2}{P_y} \cdot y + P_y \cdot y$$

$$M = y\left(\frac{P_x^2}{P_y} + P_y\right)$$

$$\frac{M}{\left(\frac{P_x^2}{P_y} + P_y\right)} = y$$

$$\frac{M \cdot P_y}{P_x^2 + P_y^2} = y^*$$

Marshallian demand function $(x^*, y^*) =$

$$\left(\frac{M \cdot P_x}{P_x^2 + P_y^2}, \frac{M \cdot P_y}{P_x^2 + P_y^2}\right) \quad \#$$

c) $\frac{dx^*}{dP_x} = \frac{(P_x^2+P_y^2)(m) - (m \cdot P_x)(2P_x)}{(P_x^2+P_y^2)^2} = -\frac{M(P_x^2-P_y^2)}{(P_x^2+P_y^2)^2}$

Law of demand → 1. x^* and y^* are positive

2. slope $= -\frac{\Delta y}{\Delta x} = \frac{-P_y}{P_x}$

3. Price of x should be more than price of y for equation

d) $y^* = \frac{M \cdot P_y}{P_x^2 +}$

$\frac{dy^*}{dP_x} = \frac{(P_x^2+P_y^2)(0) - (M \cdot P_y)(2P_x)}{(P_x^2+P_y^2)^2} = \frac{-2MP_xP_y}{(P_x^2+P_y^2)^2}$

Good x and good Y are complement products

when price of good x increase by 1 \$, demand for good y decrease by $\frac{2MP_xP_y}{(P_x^2+P_y^2)^2}$

e) $x^* = \frac{M \cdot P_x}{P_x^2+P_y^2} = \frac{300 \cdot 1}{1^2+1^2} = 150$

$y^* = \frac{M \cdot P_y}{P_x^2+P_y^2} = \frac{300 \cdot 1}{1^2+1^2} = 150$

$\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x + \lambda(-P_x) = 0$

$\lambda = \frac{x}{\sqrt{x^2+y^2}} = \frac{150}{\sqrt{150^2+150^2}} = 0.7071$

f) $\lambda^* = \frac{\Delta u}{\Delta m}$ ← 10

$\lambda^* \cdot \Delta m = \Delta u$

$0.7071(10) = \Delta u$

$7.071 = \Delta u$

Old u ; $(x^2+y^2)^{1/2} = (150^2+150^2)^{1/2} = 212.1320$

New u ; $212.1320 + 7.071 = 219.2030$

3. Suppose that a monopolist has its marginal cost function given by $MC = 16 + 6q^2$ where q is the amount of output produced. The monopolist faces the market demand function given by $P = 160 - 10q^2$ where P is the price per unit of output. Consider the following problem.

a) Suppose that fixed cost is equal to \$240. Calculate the total and the average cost when q = 9 units.

$MC = 16 + 6q^2$

$TC = \int MC\,dq$

$= \int (16 + 6q^2)\,dq$

$TC = 16q + \frac{6q^3}{3} + C$

$TC = 16q + 2q^3 + C$

$TC = 16q + 2q^3 + 240$; total cost function

when $q = 9$; $TC = 16(9) + 2(9)^3 + 240$

$= 1{,}842$

$AC = TC/q$

$= 1842/9$

$= 204.67$

∴ Total cost is \$ 1,842
average cost is \$ 204.67

b) Determine the profit-maximizing level of output for the monopolist. Also, confirm your result by using the second derivative test.

$\pi = TR - TC$ $\quad TC = 16q + 2q^3 + 240$

$= Pq - TC$ $\quad P = 160 - 10q^2$

$= (160 - 10q^2)q - (16q + 2q^3 + 240)$

$= 160q - 10q^3 - 16q - 2q^3 - 240$

$\pi = -12q^3 + 144q - 240$

$\max_q \pi = -12q^3 + 144q - 240$

FOC : $\frac{d\pi}{dq} = 0$

$-36q^2 + 144 = 0$

$-q^2 + 4 = 0$

$q^2 = 4$

$q^* = 2, \cancel{-2}$ → Profix-maximizing output of monopolist

$p^* = 160 - 10q^2$ → profit-maximizing price

$= 160 - 10(2)^2$

$p^* = 120$

SOC $\frac{d^2\pi}{dq^2} = -72q$

at $q = 2$; $-72(2) < 0$ max

∴ $q^* = 2$ that maximize profit

c) Calculate the social welfare under the monopoly environment.

d) Calculate the social welfare loss under the monopoly environment.

Demand ; $P = 160 - 10q^2$

$$\frac{dP}{dq} = -20q < 0$$

demand curve is downward sloping.

$$\frac{d^2P}{dq^2} = -20 < 0$$

demand function is concave.

c) Consumer surplus $= \int_0^{q^*} D(q)\,dq - p^* q^*$

$= \int_0^2 (160 - 10q^2)\,dq - [(120)(2)]$

$= \left[160q - \frac{10q^3}{3}\right]_0^2 - 240$

$= \left[160(2) - \frac{10}{3}(2)^3\right] - 240$

$= 53.33$

producer surplus $= p^*(q^*) - \int_0^{q^*} S(q)\,dq$

$= (120)(2) - \int_0^{q^*} MC\,dq$

$= 240 - \int_0^{q^*} (16 + 6q^2)\,dq$

$= 240 - \left[16q + \frac{6q^3}{3}\right]_0^2$

$= 240 - \left[16(2) + 2(2)^3\right]$

$= 192$

Total surplus (TS) = CS + PS

$= 53.33 + 192$

$= 245.33$

d) perfect com

$$P = MC$$

$$160 - 10q^2 = 16 + 6q^2$$

$$144 = 16q^2$$

$$9 = q^2$$

$$q^{**} = 3, \cancel{-3}$$

$$p^{**} = 160 - 10q^2 = 160 - 10(9)$$

$$p^{**} = 70$$

$$\text{deadweight loss (DWL)} = \int_{q^*}^{q^{**}} [D(q) - S(q)]\, dq$$

$$= \int_2^3 (160 - 10q^2 - 16 - 6q^2)\, dq$$

$$= \int_2^3 (-16q^2 + 144)\, dq$$

$$= \left[\frac{-16q^3}{3} + 144q \right]_2^3$$

$$= \left[\frac{-16(3)^3}{3} + 144(3) \right] - \left[\frac{-16(2)^3}{3} + 144(2) \right]$$

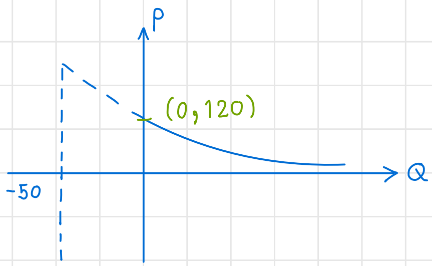
$$DWL = 288 - 245.33$$

$$= 42.67$$

4. Suppose the demand and supply curves are $P = \frac{6000}{Q+50}$ and $P = Q + 10$. Find the equilibrium price and quantity, and compute the consumer and producer surplus.

Demand equation $P = \frac{6,000}{Q+50}$ $P = 0$, $Q + 50 = 0$

$Q = -50$



Find P-intercept $(Q = 0)$

$P = \frac{6,000}{Q+50}$

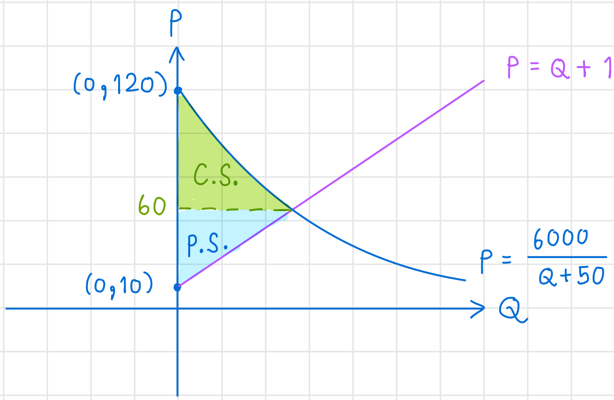
$P = 120$

supply equation $P = Q + 10$

find P-intercept $(Q = 0)$

$P = 0 + 10$

$P = 10$ $\therefore$ P-intercept $= (0, 10)$



Find equilibrium point

$P = \frac{6000}{Q+50}$ ——— ① $P = Q + 10$ ——— ②

① = ② ; $\frac{6000}{Q+50} = Q + 10$

$6000 = Q + 10(Q+50)$

$6000 = Q^2 + 50Q + 10Q + 500$

$0 = Q^2 + 60Q + 500 - 6000$

$0 = Q^2 + 60Q - 5500$

$0 = (Q+110)(Q-50)$

$Q = \sout{-110}, 50$ $\therefore Q = 50$

cannot

sub $Q = 50$ in $P = Q + 10$

$P = 50 + 10$

$P = 60$

$\therefore$ The equilibrium price = \$60 and equilibrium quantity = 50 units

consumer surplus $= \int_0^{50} \left[\frac{6,000}{Q+50}\right] - [60] \, dQ$

$= \int_0^{50} \left[\frac{6,000}{Q+50} - 60\right] dQ$

$= \left[6,000 \ln|Q+50| - 60Q\right] \Big|_{Q=0}^{Q=50}$

$$= \left[6{,}000 \ln|100| - 3{,}000\right] - \left[6{,}000 \ln|50| - 0\right]$$

$$= 6{,}000 \ln 100 - 3{,}000 - 6{,}000 \ln 50$$

$$= 6{,}000\left[\ln 100 - \ln 50\right] - 3{,}000$$

$$= 6{,}000\left[\ln\left(\frac{100}{50}\right)\right] - 3{,}000$$

consumer surplus $= 6{,}000 \ln 2 - 3{,}000$

producer surplus $= \int_0^{50} 60 - (Q + 10)\, dQ$

$$= \int_0^{50} (60 - Q - 10)\, dQ$$

$$= \int_0^{50} \left[50 - Q\right] dQ$$

$$= \left[50Q - \frac{Q^2}{2}\right] \Bigg|_{Q=0}^{Q=50}$$

$$= \left[2500 - 1250\right] - \left[0 - 0\right]$$

$$= 1{,}250$$

5. Let $MR = 25 - 5x - 2x^2$ and $MC = 10 - 3x - x^2$, where x is the unit of output. Assume that fixed cost is $7. Determine the level of production that contributes to maximum profit and determine the level of maximized profit.

หาค่า x

เอาไว้ใช้ในการแก้หา constant (c_1) ที่เกิดจากการเอา Marginal cost มาทำการ Integrate

$R = \int MR\, dx$

$R = \int [25 - 5x - 2x^2]\, dx$

$R = 25 - 5\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right) + c_1$

when $x = 0$, $R = 0$

$0 = 0 - 0 - 0 + c_1 \quad \therefore c_1 = 0$

$R = 25x - \frac{5x^2}{2} - \frac{2x^3}{3}$ —— (1)

$C = \int MC\, dx$

$C = \int (10 - 3x - x^2)\, dx$

$C = 10x - 3\left(\frac{x^2}{2}\right) - \frac{x^3}{3} + c_1$

Fixed cost = $7 when $x = 0$, $C = 7$

$7 = 0 - 0 - 0 + c_1 \quad \therefore c_1 = 7$

$C = 10x - 3\left(\frac{x^2}{2}\right) - \frac{x^3}{3} + 7$ —— (2)

Profit = revenue − total cost

$\text{Profit} = \left(25x - \frac{5x^2}{2} - \frac{2x^3}{3}\right) - \left(10x - 3\left[\frac{x^2}{2}\right] - \frac{x^3}{3} + 7\right)$

$\pi(x) = 25x - \frac{5x^2}{2} - \frac{2x^3}{3} - 10x + \frac{3}{2}x^2 + \frac{1}{3}x^3 - 7$

$\pi(x) = -\frac{1}{3}x^3 - x^2 + 15x - 7$

F.O.C : $\pi'(x) = 0$

$\pi'(x) = -\frac{3}{3}x^2 - 2x + 15$

$\pi'(x) = -x^2 - 2x + 15$ from $\pi'(x) = 0$

$0 = -x^2 - 2x + 15$

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$ critical value $x = \cancel{-5}, 3$ ($x > 0$)

So, $x = 3$ maximize the profit

S.O.C : $\pi''(x) < 0$

from $\pi'(x) = -x^2 - 25 + 15$

$\pi''(x) = -2x - 2$

$\pi''(3) = -6 - 2 = -8 < 0$

So, $x = 3$ maximize the profit

$\pi(x) = -\frac{1}{3}x^3 - x^2 + 15x - 7$

Plug in $x = 3$; $\pi(3) = -\frac{1}{3}(3)^3 - (3)^2 + 15(3) - 7$

$\pi(3) = -9 - 9 + 45 - 7$

$\pi(3) = 20$ $\therefore$ The maximum profit is $20. #