Lecture 5
CAPM

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FN 312 – INVESTMENTS

Motivations for CAPM

1. So far we have focused on the individual investor's choice problem
   - What implications does portfolio theory (our mean-variance framework) have for the market in general or in aggregate?
   - We need to make some assumptions about the conditions that must hold for the aggregate market to be in equilibrium.

2. Using the Markowitz model, the number of parameters that must be estimated is huge for active portfolio management

↪ For a portfolio of N (100) securities we need:

σ_i's	N	100
$E(r_i)$'s	N	100
$Cov(r_i, r_j)$'s	$\frac{1}{2}N(N-1)$	4950
Total	$\frac{1}{2}N(N+3)$	5150

↪ About how much data will we need when we have 500 securities? 1000 Securities?

- These parameters may be estimated with error
- Small errors in mean or covariance estimates often lead to unreasonable weights

CAPM

- A model of what returns should be will be useful
  - Recall that the standard deviation of the rate of return is not the best measure of risk for individual assets when investors hold diversified portfolios
- The CAPM is an equilibrium model specifying a relation between expected rates of return and covariances for all assets
- Note that the Markowitz portfolio optimization problem is relevant for each investor, regardless of whether the equilibrium argument, and the CAPM, is correct or not

Equilibrium Concept

- If everyone in the economy holds an efficient portfolio, then how should securities be priced so that they are actually bought 100% in equilibrium?
  - For example, if based on the prices/expected returns our model comes up with, we found that no maximizing investor would like to buy IBM, then something is wrong.
  - IBM would be priced too high
  - The price of IBM would have to fall to the point where, in aggregate, investors would want to hold exactly the number of IBM shares outstanding
- So, what sort of prices (risk/return relationships) are feasible in equilibrium?

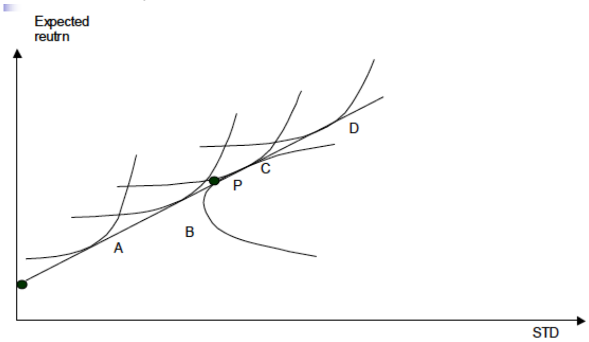
The CAPM World - Assumptions

- Perfect Competition
- No tax, transaction costs
- Rational, mean-variance optimizing agents
  - More wealth (higher expected return) is preferred to less wealth
  - Less risk (lower portfolio variance is preferred to more risk
- Investors can borrow and lend at the risk free rate
- Unlimited short sales and borrowing and lending
- Homogenous expectations
  - Investors have access to the same information and process it in the same way
- All assets are marketable, i.e. can be bought and sold and are infinitely divisible

Note: Investors may differ in risk aversion and endowments

Conceptual Derivation of CAPM

- Investors solve the Markowitz's portfolio selection problem
- Investors have the same efficient frontier and optimal CAL
- Investors hold the same optimal risky portfolio
- The proportion of wealth each investor invests in the optimal risky portfolio depends on his/her risk aversion level



Optimal risky portfolio P

w1	w2	w3	w4
0.2	0.2	0.3	0.3

Investor's Wealth	Risk Aversion	Proportion of wealth in riskfree	Proportion of risky assets	Investor' $ investment in each risky asset w1	w2	w3	w4
A: 1000	highest	0.8 ($800)	0.2 ($200)	40	40	60	60
B:10000	medium	0.04 (400)	0.96 ($9600)	1920	1920	2880	2880
C:10000	low	-0.1($-1000)	1.1 ($11000)	2200	2200	3300	3300
D: 1000	very low	-0.2 ($-200)	1.2 ($1200)	240	240	360	360
22000		**0**	**22000**	**4400**	**4400**	**6600**	**6600**

- Proportions of each security in the market portfolio = proportions of each asset in P

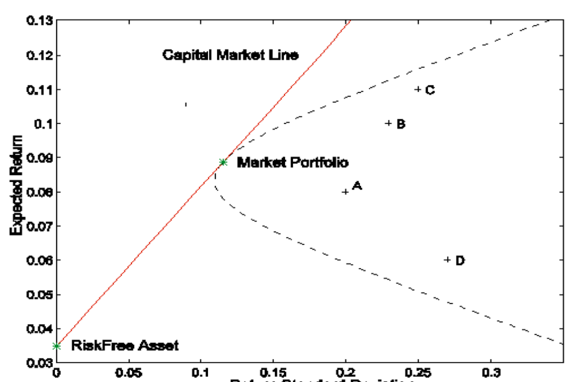
→If investors use Markowitz analysis applied to the same universe of securities, for the same horizon and input list, they must arrive at the same composition of the optimal risky portfolio

Market Portfolio

- What do we mean by the "market" portfolio?
  - Markowitz: Investors hold the tangency portfolio, so in aggregate, lending and borrowing cancel out and the average investor must be holding the market portfolio
  - CAPM: In equilibrium, every investor faces the same CAL, so the tangency portfolio must be the market portfolio

Implications of CAPM

- All investors hold the market portfolio
- The optimal CAL is the Capital Market Line (CML). This line gives us the set of efficient or optimal risk-return combinations



Implications of the CAPM (cont.)

1. The risk premium of the market portfolio is proportional to the average risk aversion of investors in the market and the variance of the market portfolio

Implications of the CAPM (cont.)

2. When there are many stocks, the expected return of individual securities depends only on its correlation with the market portfolio (beta risk)

Implications of the CAPM (cont.)

- Suppose the investor is holding the market portfolio M. The change in the variance of the market portfolio when we add asset i to the market portfolio is:
- The contribution of asset i to the risk premium of the market portfolio is

Implications of the CAPM (cont.)

- The reward to risk ratio for investment in asset i is
- The reward to risk ratio for investment in the market portfolio is

Arriving at CAPM

- Putting it all together we have the CAPM

- Beta can be interpreted as the contribution of the asset to the variance of the market portfolio
- The expected return of asset i in the market is:

Portfolio Beta

- The beta of a portfolio is a weighted average of the individual asset betas

Note: What is beta for the market portfolio?

A Formal Derivation of CAPM

- Let's do this more formally
  - Remember that, under our assumptions, all investors must hold the market portfolio
  - Based on our notion of equilibrium, every investor must be content with their portfolio holdings; if this were not the case then the prices of the securities would have to change
    - This is just a supply and demand argument; if some investors want to buy IBM, and noone wants to sell, prices will have to change (move up)
    - In equilibrium, everyone must be optimally invested
  - This must mean that, in equilbrium, noone can do anything to increase the Sharpe-ratio of their portfolio

- Suppose you currently hold the market portfolio and decide to invest in a small additional fraction of your wealth in GM, which you finance by borrowing at the risk free rate.

The return will become

$$r_C = r_m - \delta_{GM} r_f + \delta_{GM} r_{GM}$$

The expected return and variance will be

$$E(r_C) = E(r_m) + \delta_{GM}[E(r_{GM}) - r_f]$$

$$\sigma_C^2 = \sigma_m^2 + \delta_{GM}^2 \sigma_{GM}^2 + 2\delta_{GM} Cov(r_{GM}, r_m)$$

ignore δ_{GM}^2 if δ is small (0.001) then δ^2 is very small (0.00001)

The changes in each of these are

$$\Delta E(r_C) = \delta_{GM}[E(r_{GM}) - r_f]$$

$$\Delta \sigma_C^2 = 2\delta_{GM} Cov(r_{GM}, r_M)$$

- Now what if we invest more in GM, and invest just enough less in the IBM so that our portfolio variance stays the same

change in variance is

$$\Delta\sigma_c^2 = 2(\delta_{GM} \cdot cov(r_{GM}, r_m) + \delta_{IBM}\, cov(r_{IBM}, r_m))$$

To make this zero it must be the case that

$$\delta_{IBM} = -\delta_{GM}\left(\frac{cov(r_{GM}, r_m)}{cov(r_{IBM}, r_m)}\right)$$

The change in the expected return of the portfolio will be

$$\Delta E(r_c) = \delta_{GM} E(r_{GM} - r_f) + \delta_{IBM} E(r_{IBM} - r_f)$$

$$= \delta_{GM}\left[E(r_{GM} - r_f) - E(r_{IBM} - r_f)\left(\frac{cov(r_{GM}, r_m)}{cov(r_{IBM}, r_m)}\right)\right]$$

- Remember that, we are holding the market portfolio, which is also the tangency portfolio
- This portfolio has the highest Sharpe Ratio of all portfolios
- Therefore, by definition, we cannot increase its expected return while keeping variance constant
- For this to be true, it must be that

$$\frac{E(r_{GM}) - r_f}{cov(r_{GM}, r_M)} = \frac{E(r_{IBM}) - r_f}{cov(r_{IBM}, r_M)} = \lambda$$

λ = ratio of the marginal benefit to marginal cost

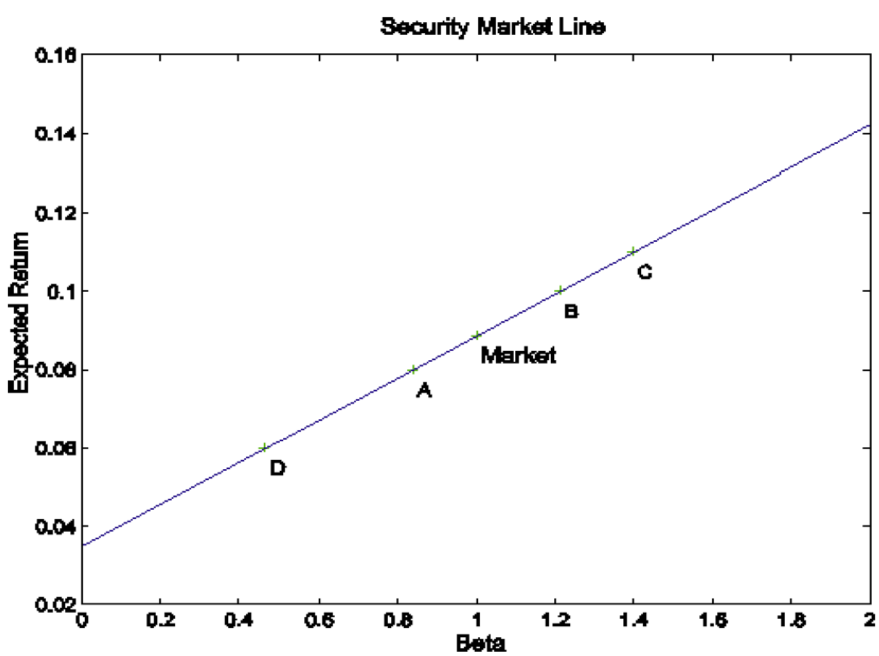
- Note that this also holds for the portfolio of assets as well
- We can use the market portfolio in place of IBM

$$\frac{E(r_{GM})-r_f}{Cov(r_{GM},r_M)} = \frac{E(r_M)-r_f}{Cov(r_M,r_m)} = \frac{E(r_M-r_f)}{\sigma^2_M} = \lambda$$

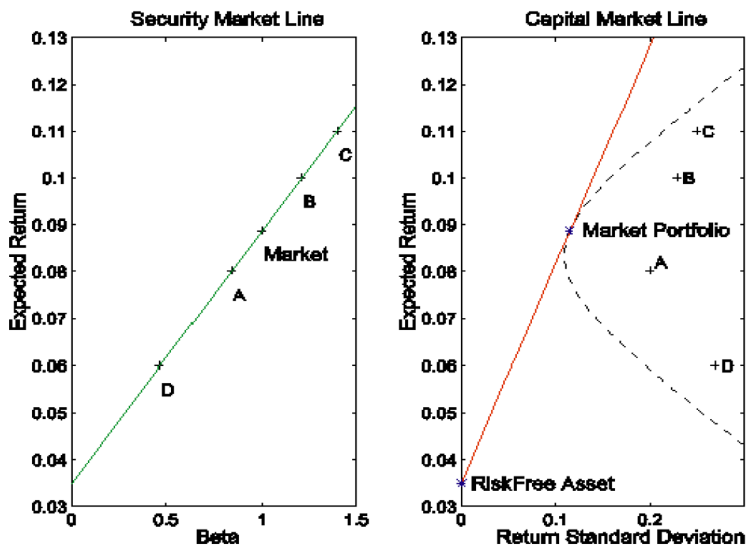
which means

$$E(r_{GM})-r_f = \frac{E(r_m-r_f)}{\sigma^2_M} cov(r_{GM},r_M)$$

$$= E(r_M-r_f) \underbrace{\frac{Cov(r_{GM},r_M)}{\sigma^2_M}}_{\beta_{GM}}$$



SML vs CML



Interpreting beta

Although an investment faces many risks, diversified investors should only care about those that are related to the market

- Zero beta
- Negative beta
- Lower beta
- Higher beta

Systematic vs Idiosyncratic Risk

Interpret beta as a regression coefficient:

- Use historical returns and estimate the time-series regression

We can decompose the total variance of returns

- Total risk = systematic market risk+unique risk
- The unique risk can be eliminated through diversification.
- Only systematic risk (captured by beta) is important in determining expected return and non systematic risk plays no role