Some economics examples:

Example: Regulated v.s. Unregulated monopolist

Suppose that a monopolist profit function is given by

$$\pi(x, y) = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

where "x" is the level of output sold to the first type of consumer and "y" is the level of output sold to the second type of consumer.

a) Find the optimal level of output that generate the highest profit

$(x^*, y^*) \rightarrow$ Constraint-free Sol^n $\rightarrow$ max π

$\boxed{\text{1-st condition:}}$

$$\frac{\partial \pi}{\partial x} = 64 - 4x + 4y = 0 \quad -①$$

$$\frac{\partial \pi}{\partial y} = 4x - 8y + 32 = 0 \quad -②$$

$① + ② \Rightarrow 96 - 4y = 0 \rightarrow y^* = 24$ units

$x \Rightarrow 4x - 8(24) + 32 = 0$

$4x = 160 \rightarrow x^* = 40$ units

Total output $\Rightarrow 40 + 24 = 64$ units.

$\boxed{\text{2-nd Condition}}$:

$$H = \begin{bmatrix} \pi_{xx} & \pi_{xy} \\ \pi_{yx} & \pi_{yy} \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 4 & -8 \end{bmatrix}$$

$|H_1| = -4 < 0$; $|H_2| = 16 > 0 \rightarrow H$: negative def.

$d^2\pi < 0$

↳ π is Concave.

$\pi^*(40, 24) =$ ←

↳ $(40, 24) \rightarrow$ max

$$\boxed{\text{Bordered Hessian Matrix}} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{\lambda y} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$

$$= \begin{array}{ccc|cc} & & +4 & 0 & +8 \\ 0 & -1 & -1 & 0 & -1 \\ -1 & -4 & 4 & -1 & -4 \\ -1 & 4 & -8 & -1 & 4 \\ & & 0 & +4 & +4 \end{array}$$

$$\Rightarrow \bar{H} = 4 + 0 + 8 + 0 + 4 + 4 = \boxed{20}$$

→ Sign is greater than zero

→ matches with the sufficient condition for the maximum Sol^{n}.

b) What happen if the government limits the total of quantity output to be equal to 79 units.

$\hookrightarrow x+y=79 \rightarrow$ Regulatory Constraint imposed!

$$\max_{(x,y)} \ \pi(x,y)$$
$$\text{s.t. } x+y=79$$

LaGrange Method.

$$\mathcal{L}(x,y,\lambda) = 64x-2x^2+4xy-4y^2+32y-14 + \lambda[79-x-y]$$

(x,y,λ) $\mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_\lambda = 0$

$\mathcal{L}_x = 64-4x+4y-\lambda = 0$ —①

$\mathcal{L}_y = 4x-8y+32-\lambda = 0$ —②

$\mathcal{L}_\lambda = 79-x-y = 0$ —③ $\rightarrow y = 79-x$

①-②; $32-8x+12y=0$ —④

$32-8\cdot x+12(79-x)=0$

$32+12\cdot 79 = 20x \rightarrow x^* = 49$ units.

$\lambda^* = -12$ $y^* = 30$ units

c) If the government allows for the production limit to be 80 units, what would happen to the optimized level of profit?

$\pi^*(79) \Rightarrow$ one thing

$\pi^*(80) \downarrow$

$x + y = 79$

$c_0 = 1$

$c_1 = 1$

$c_2 = 79$

$$\frac{d\pi^*}{dc_2} = \lambda^*$$

$$= -12$$

$\therefore$ $c_2 \uparrow$ 79 → 80 units drop by.

$\pi^*(79) \longrightarrow \pi^*(80)$ by 12\$

Example: **Utility maximization problem** Budget: $P_x . X + P_y \cdot y = M$

Suppose that a household has the preference relationship defined by the utility function $U(x,y) = 2x^{0.6}y^{0.3}$. Suppose further that the price of goods x is P_x and the price of goods y is P_y. Given M as the budget of this household, find the optimal bundle of consumption for goods x and goods y.

$$\mathcal{L} = \boxed{2\cdot x^{0.6} y^{0.3}} + \lambda \left[M - P_x\cdot x - P_y y\right]$$

1-st

$$\mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_\lambda = 0 \Rightarrow MU_x$$

$$\mathcal{L}_x = \underline{2\cdot(0.6)x^{-0.4}\cdot y^{0.3}} - \lambda\cdot P_x = 0$$

$= MU_y$

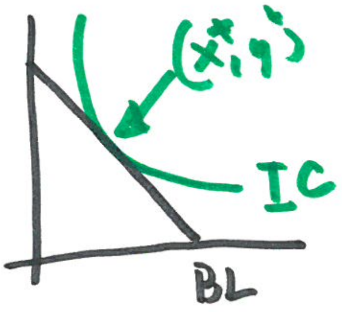
$$\mathcal{L}_y = 2(0.3)x^{0.6}y^{-0.7} - \lambda\cdot P_y = 0$$

$$\mathcal{L}_\lambda = M - P_x\cdot x - P_y\cdot y = 0$$

$$\left.\frac{MU_x}{MU_y} = \frac{P_x}{P_y}\right\} \text{MRS} = \text{Price ratio.}$$

Tangentcy condition holds

$\underline{IC}$ tangent to $\underline{BL}$



Slope IC = slope BL

$$\frac{MUx}{MUy} = \frac{2(0.6)x^{-0.4}y^{0.3}}{2(0.3)x^{0.6}y^{-0.7}} = \frac{P_x}{P_y}$$

Tangency Condition

$$y = \frac{1}{2}\cdot\frac{P_x}{P_y}\cdot x \leftarrow 2\cdot\left(\frac{y}{x}\right) = \frac{P_x}{P_y} \quad ④$$

$$P_x\cdot x + P_y\cdot y = M$$

$$P_x\cdot x + \cancel{P_y}\left(\frac{1}{2}\frac{P_x}{\cancel{P_y}}\cdot x\right) = M$$

$$P_x\cdot x + \frac{1}{2}\cdot P_x\cdot x = M$$

$$\frac{3}{2}P_x\cdot x = M$$

Marshallian Demand for x

$$\rightarrow x = \frac{2}{3}\cdot\frac{M}{P_x}$$

utility-maximizing bundle of good x

Suppose $M = 100$; $P_x = 2 \Rightarrow x = \frac{2}{3}\left(\frac{100}{2}\right) = \frac{100}{3}$ units

$$y = \frac{1}{2}\cdot\frac{P_x}{P_y}\cdot x = \frac{1}{2}\cdot\overset{x}{\left(\frac{2}{3}\cdot\frac{M}{P_x}\right)}\cdot\frac{P_x}{P_y} = \frac{1}{3}\cdot\frac{M}{P_y}$$

utility-maximizing bundle of good y

Exercise: Redo the example with the new utility function $U(x,y) = (x-1)^{\frac{1}{2}}(y-4)^{\frac{1}{2}}$.

D for x & D. for y is HM°; no money illusion

M↑, P_x↑, P_y↑ proportionately → x, y same

The obtained solutions, $x^*(p_x, p_y, M)$ *and* $y^*(p_x, p_y, M)$, from the above utility maximization problem (UMP) yield us the so called "*Marshallian demand function*". The function is HM degree zero in prices and income.

Indirect utility function: = optimized level of utility f^n of HH

The optimized level of utility can be derived by plugging the optimal consumption bundle into the utility function. Economically, the optimized utility function, expressed in terms of prices and income, is called the ***indirect utility function***. Mathematically, the indirect utility function is the optimal value function attained from the optimization problem.

$$u(x,y) = 2 \cdot X^{0.6} \cdot y^{0.3} \Rightarrow 2\left(\frac{2}{3} \cdot \frac{M}{P_x}\right)^{0.6}\left(\frac{1}{3} \cdot \frac{M}{P_y}\right)^{0.3}$$

$$v(p_x, p_y, M) = U\left(\boldsymbol{x}^*(\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{M}), \boldsymbol{y}^*(\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{M})\right)$$

$$\Downarrow$$

$$v(P_x, P_y, M)$$

Properties:

(i) HM degree of zero in prices and income.

a) $P_x\uparrow = P_y\uparrow = M\uparrow \rightarrow x, y$ to remain the same

Proof: (b) $u(x^*, y^*)$ would remain the same as before.

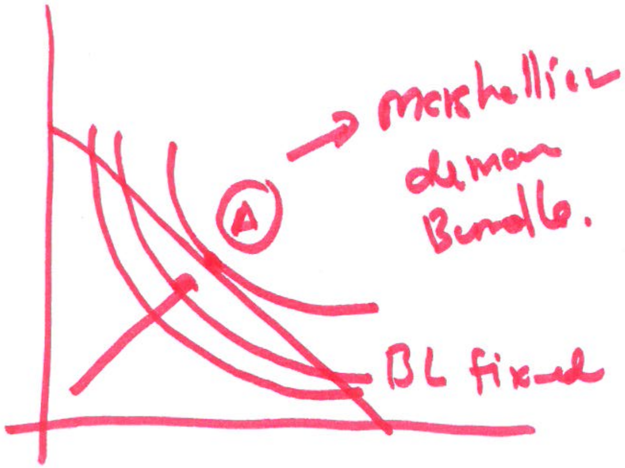
(ii) $\boldsymbol{x}^* = -\dfrac{\frac{\partial v}{\partial p_x}}{\frac{\partial v}{\cancel{\partial p_M}\,\partial M}}$ and $\boldsymbol{y}^* = -\dfrac{\frac{\partial v}{\partial p_y}}{\frac{\partial v}{\cancel{\partial p_M}\,\partial M}}$ <u>(Roy's identity)</u>

Proof: Recoverability ⇒ Recover x^*, y^* from $\underline{\underline{v}}$

$$x^* = -\frac{\frac{\partial v}{\partial P_x}}{\frac{\partial v}{\partial M}} \qquad \frac{\partial v}{\partial P_x} = \frac{\partial \mathcal{L}}{\partial P_x} = -\lambda \cdot x = -X$$

$$\frac{\partial v}{\partial M} = \frac{\partial \mathcal{L}}{\partial M} = \lambda$$

UMP



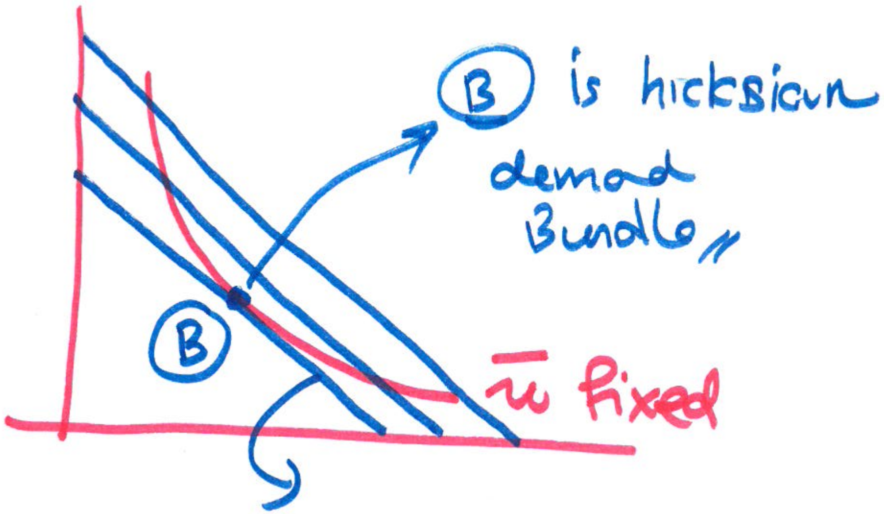
$\max \quad u(x,y)$

$\text{s.t.} \quad P_x x + P_y y = M$

$\hookrightarrow \quad x^*(P_x, P_y, M)$

EMP

Expenditure $f^n \Rightarrow P_x \cdot x^h + P_y \cdot y^h$



Iso expenditure line

Iso cost curve.

$\min \quad P_x X + P_y Y$

$\text{s.t.} \quad \cancel{P} \; u(x,y) = \bar{u}$

$X^h(P_x, P_y, \bar{u})$

Compensated demand function (Hicksian demand curve)

If you studied EE311, you had seen this concept before, the Hicksian demand curve. Expenditure Minimization Problem (EMP)

An alternative form of the demand function can be derived from the *optimal bundle that minimizes the level of expenditure.* Hicks introduced this concept after Marhsall's introduction on demand function, and it has become known since then under the name of the *compensated demand function*.

Hicks's idea is that, instead of searching for the bundle that maximize the utility under budge, household is then assumed to choose the bundle that minimizes the level of expenditure needed for acquiring a certain level of utility. The problem under Hicks's supposition is then

$$mix\ \boxed{p_x x + p_y y}\ s.t.\, u(x, y) = \bar{u}$$

Solution to the above Hicks's problem can be derived in a similar way to the problem stated out by Marshall. That, is we use the LaGrange method.

The resulting optimization problem yields us the solution where

$$x^h = x^h(p_x, p_y, \bar{u}) \text{ and } y^h = y^h(p_x, p_y, \bar{u})$$

Given the Hicksian demand bundle, the optimized level of expenditure, commonly called the expenditure function, is given by,

Plugging x^h, y^h into the isoexpenditure f^2

$$\underline{e(p_x, p_y, \bar{u}) = p_x x^h(p_x, p_y, \bar{u}) + p_y y^h(p_x, p_y, \bar{u})}$$

Duality in Household decision problem

(i) $$x^h\left(p_x, p_y, v(p_x, p_y, M)\right) = x^*(p_x, p_y, M)$$

(ii) $$x^*\left(p_x, p_y, e(p_x, p_y, \bar{u})\right) = x^h(p_x, p_y, \bar{u})$$

(iii) Roy's Identity:

$$x^* = -\frac{\frac{\partial v}{\partial p_x}}{\frac{\partial v}{\partial p_M}} \text{ and } y^* = -\frac{\frac{\partial v}{\partial p_y}}{\frac{\partial v}{\partial p_M}}$$

(iv) Expenditure function:

$$x^h = \frac{\partial e(p_x, p_y, \bar{u})}{\partial p_x} \quad \text{and} \quad y^h = \frac{\partial e(p_x, p_y, \bar{u})}{\partial p_y}$$

Doing K, L → max Π → Set-up results in optimal K, L ⇒ Demand for K, L

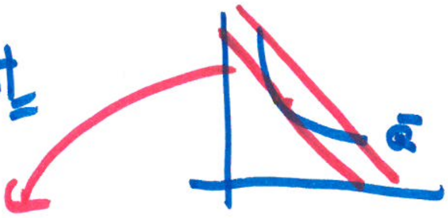
Unconditional

Example: **Cost function**

Suppose that firm has the production technology given by $Q = \sqrt{K} + \sqrt{L}$. To acquire each unit of capital (K) and labor (L), firm must pay for the unit cost of r and w, respectively.

a) Derive the optimal combinations of factor inputs. that ~~minim~~ minimize firm's cost

- Conditional factor inputs demand.
- ~~Isoque~~ ~~Isq~~ Isoquant



red line: Iso Cost Curve.

⇒ $w \cdot L + r \cdot K$ → Iso Cost equation

$$\min_{K,L} \quad wL + rK$$

$$\text{s.t.} \quad Q = \sqrt{K} + \sqrt{L}$$

Set up the LaGrangf^n

$$\mathcal{L} = wL + rK + \lambda(Q - \sqrt{K} - \sqrt{L})$$

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda \cdot \frac{1}{2\sqrt{L}} = 0 \quad -①$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \cdot \frac{1}{2 \sqrt{K}} = 0 \quad -②$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Q - \sqrt{K} - \sqrt{L} = 0 \quad ③$$

$$\frac{②}{①}\;; \quad \frac{r}{w} = \frac{\sqrt{L}}{\sqrt{K}} \Rightarrow \frac{L}{K} = \left(\frac{r}{w}\right)^2$$

$$L = \left(\frac{r}{w}\right)^2 \cdot K$$

$$Q = \sqrt{K} + \frac{r}{w} \cdot \sqrt{K} \Rightarrow \frac{w\sqrt{K} + r\sqrt{K}}{w} = Q$$

$$\sqrt{K} = \frac{w \cdot Q}{w+r} \Rightarrow K^* = \left[\frac{w}{w+r} \cdot Q\right]^2$$

↳ optimal factor input for k that minimizes cost of producing Q units; i.e. conditional factor demand for capital (k).

$$L^* = \left[\frac{r}{w+r} \cdot Q\right]^2$$

$$\frac{\partial K^*}{\partial Q} = 2\left(\frac{w}{w+r} \cdot Q\right)\left(\frac{w}{w+r}\right) > 0$$

$$\frac{\partial L^*}{\partial Q} = 2\left(\frac{r}{w+r} \cdot Q\right)\left(\frac{r}{w+r}\right) > 0$$

$$\frac{\partial K^*}{\partial r} < 0 \qquad \underline{\frac{\partial K^*}{\partial w} > 0}.$$

$$\frac{\partial L^*}{\partial w} < 0$$

$\underline{\underline{Q\uparrow}}$ → more factor inputs needed for the <u>CMP</u>

b) What is the cost function?

Cost f^n: ≠ Iso Cost equation

Definition describing the cost of production $wL + rK$

↓

minimized level of the Iso cost equation.

$$C = w \cdot L^* + r \cdot K^*$$

$$C = w \cdot \left(\frac{r}{w+r} \cdot Q\right)^2 + r\left(\frac{w}{w+r} \cdot Q\right)^2$$

$\underline{C}(w, r, \underline{Q})$: optimal Cost f^n

Long-run Cost f^n

$\frac{\partial C^*}{\partial Q}$ = marginal Cost ←

□

$$\frac{\partial C}{\partial Q} = \frac{\partial \mathcal{L}}{\partial Q} = \lambda^* \longrightarrow \triangle$$

□ = △