15. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \quad \Rightarrow \quad h = (1200 - b^2)/(4b)$.

The volume is $V = b^2h = b^2(1200 - b^2)/4b = 300b - b^3/4 \quad \Rightarrow \quad V'(b) = 300 - \frac{3}{4}b^2$.

$V'(b) = 0 \quad \Rightarrow \quad 300 = \frac{3}{4}b^2 \quad \Rightarrow \quad b^2 = 400 \quad \Rightarrow \quad b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 253).

If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2(10) = 4000\text{ cm}^3$.

57. (a) If $c(x) = \dfrac{C(x)}{x}$, then, by the Quotient Rule, we have $c'(x) = \dfrac{xC'(x) - C(x)}{x^2}$. Now $c'(x) = 0$ when

$xC'(x) - C(x) = 0$ and this gives $C'(x) = \dfrac{C(x)}{x} = c(x)$. Therefore, the marginal cost equals the average cost.

(b) (i) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so

$C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$,

$C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

(ii) We must have $C'(x) = c(x) \quad \Leftrightarrow \quad 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \quad \Leftrightarrow \quad 2x^{3/2} = 16{,}000 \quad \Leftrightarrow$

$x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate

$c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}(x^{3/2} - 8000)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$,

and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(iii) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

61. (a) As in Example 6, we see that the demand function p is linear. We are given that $p(1000) = 450$ and deduce that

$p(1100) = 440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is $\frac{440 - 450}{1100 - 1000} = -\frac{1}{10}$,

so an equation is $p - 450 = -\frac{1}{10}(x - 1000)$ or $p(x) = -\frac{1}{10}x + 550$.

(b) $R(x) = xp(x) = -\frac{1}{10}x^2 + 550x$. $R'(x) = -\frac{1}{5}x + 550 = 0$ when $x = 5(550) = 2750$.

$p(2750) = 275$, so the rebate should be $450 - 275 = \$175$.

(c) $C(x) = 68{,}000 + 150x \quad \Rightarrow \quad P(x) = R(x) - C(x) = -\frac{1}{10}x^2 + 550x - 68{,}000 - 150x = -\frac{1}{10}x^2 + 400x - 68{,}000$,

$P'(x) = -\frac{1}{5}x + 400 = 0$ when $x = 2000$. $p(2000) = 350$. Therefore, the rebate to maximize profits should be

$450 - 350 = \$100$.

17. Let $w = 1 - 3x$. Then $\dfrac{dw}{dx} = -3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{dw}\dfrac{dw}{dx}$, so

$$y' = \frac{d}{dx}\int_{1-3x}^{1} \frac{u^3}{1+u^2}\,du = \frac{d}{dw}\int_{w}^{1} \frac{u^3}{1+u^2}du \cdot \frac{dw}{dx} = -\frac{d}{dw}\int_{1}^{w} \frac{u^3}{1+u^2}\,du \cdot \frac{dw}{dx} = -\frac{w^3}{1+w^2}(-3) = \frac{3(1-3x)^3}{1+(1-3x)^2}$$

28. $\int_0^1 (3 + x\sqrt{x}\,)\,dx = \int_0^1 (3 + x^{3/2})\,dx = \left[3x + \frac{2}{5}x^{5/2}\right]_0^1 = \left[\left(3 + \frac{2}{5}\right) - 0\right] = \frac{17}{5}$

30. $\int_0^2 (y-1)(2y+1)\,dy = \int_0^2 (2y^2 - y - 1)\,dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^2 - y\right]_0^2 = \left(\frac{16}{3} - 2 - 2\right) - 0 = \frac{4}{3}$

34. $\displaystyle\int_1^2 \frac{s^4+1}{s^2}\,ds = \int_1^2 (s^2 + s^{-2})\,ds = \left[\frac{1}{3}s^3 - \frac{1}{s}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right) = \frac{7}{3} + \frac{1}{2} = \frac{17}{6}$

40. $f(x) = \dfrac{4}{x^3}$ is not continuous on the interval $[-1, 2]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\displaystyle\int_{-1}^{2} \frac{4}{x^3}\,dx$ does not exist.

50. $\displaystyle g(x) = \int_{1-2x}^{1+2x} t\sin t\,dt = \int_{1-2x}^{0} t\sin t\,dt + \int_{0}^{1+2x} t\sin t\,dt = -\int_{0}^{1-2x} t\sin t\,dt + \int_{0}^{1+2x} t\sin t\,dt \quad\Rightarrow$

$$\begin{aligned} g'(x) &= -(1-2x)\sin(1-2x)\cdot\frac{d}{dx}(1-2x) + (1+2x)\sin(1+2x)\cdot\frac{d}{dx}(1+2x) \\ &= 2(1-2x)\sin(1-2x) + 2(1+2x)\sin(1+2x) \end{aligned}$$

14. Let $x = 1 - u^2$. Then $dx = -2u\,du$ and $u\,du = -\frac{1}{2}\,dx$, so

$\int u\sqrt{1-u^2}\,du = \int \sqrt{x}\left(-\frac{1}{2}\,dx\right) = -\frac{1}{2}\int x^{1/2}\,dx = -\frac{1}{2}\cdot\frac{2}{3}x^{3/2} + C = -\frac{1}{3}(1-u^2)^{3/2} + C.$

16. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ and $2\,du = \dfrac{1}{\sqrt{x}}\,dx$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) = -2\cos u + C = -2\cos\sqrt{x} + C.$$

24. Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$, so

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} = \int \frac{\sec^2 t\,dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

30. Let $u = x^2 + 1$ [so $x^2 = u - 1$]. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u-1)\sqrt{u}\,\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)du$$
$$= \tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right) + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C.$$

Or: Let $u = \sqrt{x^2+1}$. Then $u^2 = x^2 + 1 \;\Rightarrow\; 2u\,du = 2x\,dx \;\Rightarrow\; u\,du = x\,dx$, so

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u^2-1)\,u\cdot u\,du = \int \left(u^4 - u^2\right)du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C.$$

Note: This answer can be written as $\frac{1}{15}\sqrt{x^2+1}\,(3x^4 + x^2 - 2) + C$.

48. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u-1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\int_0^4 \frac{x\,dx}{\sqrt{1+2x}} = \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\frac{du}{2} = \frac{1}{4}\int_1^9 \left(u^{1/2} - u^{-1/2}\right)du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9 = \frac{1}{4}\cdot\frac{2}{3}\left[u^{3/2} - 3u^{1/2}\right]_1^9$$
$$= \tfrac{1}{6}[(27-9) - (1-3)] = \tfrac{20}{6} = \tfrac{10}{3}$$

69. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C$.

82. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e)$.