

Macroeconomics

Lecture 2

A Recursive Problem

- Hence, one can continue recursively in this way, solving for a collection of feedback rules of the form

$$c_t = h_t(x_t), \quad t = T, T-1, T-2, \dots, 0.$$

where both $c_t = h_t(x_t)$ and $x_{t+1} = f_t(x_t)$ solve these equations

$$\frac{\partial U_t(x_t, c_t)}{\partial c_t} + \frac{\partial g_t(x_t, c_t)}{\partial c_t} \left\{ \frac{\partial U_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}} \left[\frac{\partial U_{t+2}}{\partial x_{t+2}} + \frac{\partial g_{t+2}}{\partial x_{t+2}} \left\{ \frac{\partial U_{t+3}}{\partial x_{t+3}} + \frac{\partial g_{t+3}}{\partial x_{t+3}} \left\{ \dots + \frac{\partial g_{T-1}}{\partial x_{T-1}} \left[\frac{\partial U_T}{\partial x_T} \right] \right\} \dots \right] \right\} \right\} = 0,$$

for $t = 0, 1, \dots, T.$

A Recursive Problem

and

$$x_{s+1} = g_s(x_s, c_s) \quad \text{for } s = t, t+1, \dots, T.,$$

given that:

$$c_{s+1} = h_{s+1}(x_{s+1}) \quad \text{for } s = t, t+1, \dots, T-1.$$

Bellman's Equations

- Define the value function for a one-period problem

$$W_1(x_T) = \max_{c_T} \{U_T(x_T, c_T)\} \quad (1.6)$$

Subject to

$$x_{T+1} = g_T(x_T, c_T),$$

with x_T given

Bellman's Equations

- Form the Lagrangian, find 1st-order conditions,

- $$\frac{\partial U_T(x_T, c_T)}{\partial c_T} \geq 0 \quad (1.7)$$

- Which is the same as equation (1.5c).
- Equation (1.7) and $x_{T+1} = 0 = g_T(x_T, c_T)$ are jointly determined $c_T = h_T(x_T)$ and $x_{T+1} = 0 = g(x_T, h_T(x_T))$. Then, put this outcome into (1.6), one has

- $$W_1(x_T) = U_T[x_T, h_T(x_T)] \quad (1.8)$$

Bellman's Equations

- Differentiating (1.8), w.r.t. x_T , gives

$$\frac{\partial W_1(x_T)}{\partial x_T} = W_1'(x_T) = \left(\frac{\partial U_T}{\partial x_T} \right) + \frac{\partial h_T}{\partial x_T} \left[\frac{\partial U_T}{\partial h_T} \right]$$

- Then, together with (1.7), one has

- $$W_1'(x_T) = \frac{\partial U_T(x_T, h_T(x_T))}{\partial x_T} + \frac{\partial h_T}{\partial x_T} \left[\frac{\partial U_T(\cdot)}{\partial c_T} \right] \quad (1.9)$$

Bellman's Equations

- Next, repeat the process with the value function for the two-period problem,

- $$W_2(x_{T-1}) = \max_{c_{T-1}} \{U_{T-1}(x_{T-1}, c_{T-1}) + W_1(x_T)\} \quad (1.10)$$

- where
$$W_1(x_T) = \max_{c_T} \{U_T(x_T, c_T)\}$$

- subject to

$$x_T = g_{T-1}(x_{T-1}, c_{T-1}),$$

with x_{T-1} given

Bellman's Equations

- Again, by repeating the same procedure, we have

- $$W_2'(x_{T-1}) = \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} + \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial h_{T-1}(x_{T-1})} \cdot \frac{\partial h_{T-1}(x_{T-1})}{\partial x_{T-1}}$$
- $$+ \frac{\partial g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} W_1'(g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1})))$$
 (1.11)

- Applying (1.9) into (1.11), one has

- $$W_2'(x_{T-1}) = \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} + \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial h_{T-1}(x_{T-1})} \cdot \frac{\partial h_{T-1}(x_{T-1})}{\partial x_{T-1}}$$
- $$+ \frac{\partial g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} \cdot \left\{ \frac{\partial U_T}{\partial x_T} + \frac{\partial h_T}{\partial x_T} \left[\frac{\partial U_T}{\partial c_T} \right] \right\},$$
 (1.12)

- where $x_T = g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))$

Bellman's Equations

- Note that, for $j = 1$, equation (1.12) becomes

$$W'_{j+1}(x_{T-j}) = \frac{\partial U_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))}{\partial x_{T-j}} + \frac{\partial U_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))}{\partial h_{T-j}(x_{T-j})} \cdot \frac{\partial h_{T-j}(x_{T-j})}{\partial x_{T-j}} + \frac{\partial g_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))}{\partial x_{T-j}} \cdot \left\{ \frac{\partial U_T}{\partial x_T} + \frac{\partial h_T}{\partial x_T} \frac{\partial U_T}{\partial c_T} \right\}, \quad (1.13)$$

where

$$x_{T-j+1} = g_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))$$

Bellman's Equations

- By iteration, one has the Bellman's equation as

- $$W_{j+1}(x_{T-j}) = \max_{c_{T-j}} \left\{ U_{T-j}(x_{T-j}, c_{T-j}) + W_j(x_{T-j+1}) \right\} \quad (1.14)$$

- subject to

$$x_{T-j+1} = g_{T-j}(x_{T-j}, c_{T-j}),$$

with x_{T-j} given

Bellman's Equations

- The idea is to proceed recursively and to work backward.
- First solve the 1-period problem with $j+1=T$, deducing $W_1(x_T)$. (or eq.(1.6))
- Then solve the 2-period problem with $j+1=T-1$, deducing $W_2(x_{T-1})$. (or eq.(1.10))
- The process is repeated until we have the $(T+1)$ -period value function, $W_{T+1}(x_0)$, and along the way we have the optimal feedback rules $c_{T-j}=h_{T-j}(x_{T-j}), j = 0, 1, \dots, T$.

Bellman's Equations

- The $(T+1)$ -period value function is;

$$\begin{aligned} W_{T+1}(x_0) &= \max_{c_0} \left[U_0(x_0, c_0) + \max_{c_1} \left[U_1(x_1, c_1) + \dots \max_{c_T} \left[U_T(x_T, c_T) \right] \right] \right] \\ &= \max_{c_0} \left[U_0(x_0, c_0) + W_T(x_1) \right] \end{aligned}$$

Subject to $x_1 = g_0(x_0, c_0)$, x_0 given.

Bellman's Equations

- The first-order condition of the above problem w.r.t. u_0 is,

$$\frac{\partial U_0(x_0, c_0)}{\partial c_0} + \frac{\partial g_0(x_0, c_0)}{\partial c_0} W_T'(x_1) = 0 \quad (1.15)$$

- Rewrite equation(1.14) for $j = T-1$, and put into eq.(1.15), we have

$$\frac{\partial U_t(x_t, c_t)}{\partial c_t} + \frac{\partial g_t(x_t, c_t)}{\partial c_t} \left\{ \frac{\partial U_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}} \left[\frac{\partial U_{t+2}}{\partial x_{t+2}} + \frac{\partial g_{t+2}}{\partial x_{t+2}} \left\{ \frac{\partial U_{t+3}}{\partial x_{t+3}} + \frac{\partial g_{t+3}}{\partial x_{t+3}} \left\{ \dots + \frac{\partial g_{T-1}}{\partial x_{T-1}} \left[\frac{\partial U_T}{\partial x_T} \right] \right\} \dots \right] \right\} \right\} = 0,$$

for $t = 0, 1, \dots, T.$

Bellman's Equations

- This backward recursion generates the same marginal conditions as the problem (1.5a)
- The derivation of the value functions obeys the recursion (see eq.(13) & eq. (1.9), for $j=1$)

$$W'_{j+1}(x_{T-j}) = \frac{\partial U_{T-j}[x_{T-j}, h_{T-j}(x_{T-j})]}{\partial x_{T-j}} + \frac{\partial U_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))}{\partial h_{T-j}(x_{T-j})} \cdot \frac{\partial h_{T-j}(x_{T-j})}{\partial x_{T-j}} + \frac{\partial g_{T-j}}{\partial x_{T-j}} W'_j(g_{T-j}[x_{T-j}, h_{T-j}(x_{T-j})])$$

- Comparing this equation with (1.4bbb) and (1.4ccc), we find that:

Bellman's Equations

- $W'_j(x_{T+1-j}) = \lambda_{T-j}$,
- and $W'_{j+1}(x_{T-j}) = \lambda_{T-j-1}$,
- Hence,

$$\lambda_{T-(j+1)} = \frac{\partial U_{T-j} [x_{T-j}, h_{T-j}(x_{T-j})]}{\partial x_{T-j}} + \frac{\partial U_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))}{\partial h_{T-j}(x_{T-j})} \cdot \frac{\partial h_{T-j}(x_{T-j})}{\partial x_{T-j}} + \frac{\partial g_{T-j}}{\partial x_{T-j}} \lambda_{T-j}$$

Summary

- Define the (T+1)-period value function

$$W_{T+1}(x_0) = \max_{c_0, \dots, c_T} [U_0(x_0, c_0) + U_1(x_1, c_1) + \dots + U_T(x_T, c_T)] \quad (1.16)$$

Subject to $x_{t+1} = g_t(x_t, c_t)$, for all $t = 0, 1, \dots, T$; x_0 given.

Problem (1.16) is the same as

$$\begin{aligned} W_{T+1}(x_0) &= \max_{c_0} \left[U_0(x_0, c_0) + \max_{c_1} \left[U_1(x_1, c_1) + \dots \max_{c_T} [U_T(x_T, c_T)] \right] \right] \quad (1.17) \\ &= \max_{c_0} [U_0(x_0, c_0) + W_T(x_1)] \end{aligned}$$

subject to $x_1 = g_0(x_0, c_0)$, x_0 given.

Summary

Starting to solve problem (1.16) from $t = T$,

- (1) Solving for $c_T = h_T(x_T)$ to optimize $W_1(x_T)$;
- (2) Solving for $c_{T-1} = h_{T-1}(x_{T-1})$ to optimize $W_2(x_{T-1})$;
- (3) Continue to repeat this process until $t=0$.