Lecture on introductory Mathematical Economics

EE320

Semester 1/2017

Part II

Multivariate analysis and Mathematical optimization in economics

BE International program

Faculty of Economics

Thammasat University

Instructor: Kittichai Saelee

Email: kittichai_lee@econ.tu.ac.th

Chapter 6: Function of several variables and Multivariate calculus

6.1) Function of several variables

Suppose we have two variables (x, y). A function that maps these two variables into another set valued of variable, z, is called function of two variables.

Example 6.A $f(x, y) = 2x + x^2y^3$ $z = x^{\frac{1}{2}} * y^{\frac{1}{2}}$

Similarly, a function that maps from the value of N-variable into another set valued of variable, z, is called function of N variables.

Example 6.B $f(x_1, x_2, \dots x_n) = x_1x_2x_3 \dots x_n$

Most economic functions require the representation of multivariate relationship.

Utility function: $U = f(x_1, x_2, x_3, \dots, x_n)$

Production function: y = f(K,L,…)

Demand function: $Q_x^d = f(P_x, I, P_y, T)$

Supply function: $Q_x^s = f(P_x, w, P_y)$

6.2) Multivariate differential calculus

6.2.1) Partial derivative

Definition: Suppose $y = f(x_1, x_2, x_3, \ldots, x_n)$, i.e. $f: R^n \to R$, the partial derivative of "y" with respect to "x_i" is denoted by:

$$\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_i \quad = \lim_{h \to 0} \frac{f\left(x_1, \ \ldots, \ x_i + h, \ldots, x_n\right) - f(x_1, \ \ldots, x_i, \ldots, x_n)}{h}$$

- Operationally, deriving the partial derivative of a function is so simple!!
- We simply treat all other variables, except x_i, as constant.
- Then, apply all the rule of differentiations that we knew from the case of single variable calculus to the multivariate function.
- Let's proceed to some examples.

Example 6.C:

Suppose that $U = -5x^2 - 12xy - 6y^2$ find $\frac{\partial U}{\partial x}$ $(= U_x)$ and $\frac{\partial U}{\partial y}$ $(= U_y)$.

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Graphical illustration of partial derivative

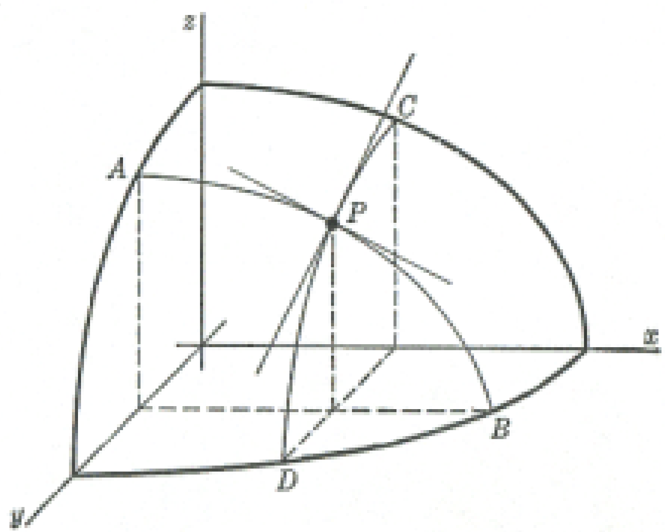


Fig. 1

Definition: *Gradient vector*

Vector of the Collection of all the first-order partial derivative terms in matrix (vector)

$$\nabla U = \begin{bmatrix} U_x \\ U_y \end{bmatrix}$$

For any arbitrary "n" variables, the gradient vector of $y = f(x_1, x_2, x_3, \ldots, x_n)$ is given by

$$\nabla f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}_{n\times 1}$$

Example 6.D: $f(x_1, x_2, \ldots x_n) = x_1 x_2 x_3 \ldots x_n$

Higher-order partial derivative

- Second-order partial derivative: $\left(\frac{\partial^2 f}{\partial x_i^2}\right)$

Differentiating the first-order derivative function with the same argument: $\frac{\partial\left(\frac{\partial f}{\partial x_i}\right)}{\partial x_i}$, i.e. differentiating the multivariate function with respect to the ***same*** argument twice.

- Second-order cross partial derivative: $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \frac{\partial\left(\frac{\partial f}{\partial x_i}\right)}{\partial x_j} = f_{ij}$

What does the second-order partial derivative tell us?

- Concavity/Convexity along a certain axis!

Example 6.C (cont): What is the second-order partial derivatives of the function used in Example 6.C above

Definition:

Hessian matrix: Collection of the second-order partial derivative in matrix form

$$H = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix}$$

Example 6.C (cont): Hessian matrix of U

Notice an important property of the Hessian matrix

For any arbitrary "n" variables, the gradient vector of $\mathrm{y} = f(x_1, x_2, x_3, \dots, x_n)$ is given by

$$H = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix}_{n\times n}$$

Example 6.D (cont): Hessian matrix of "f"

<u>What does partial derivative tell us in economics?</u>

- Economically, what does the partial derivative, i.e. $\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_i \ (x_1, x_2, x_3, \dots, x_n)$, tell us about?
- Marginal effect of x_i on y. (marginal treatments)
  - Measuring the change in "y" that is purely attributed to the change in particular "x", while keeping all other "x" stay the same.
- This basically captures the concept so called "*Ceteris Paribus*".

Example 6.E Consider a production function given by, $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$

a) Derive the expression for marginal product of labor (MPL) and capital (MPK), respectively

b) How does the MPK change with respect to k? Similarly, how does the MPL change with respect to L?

c) How does the MPL change with respect to k? Discuss the implication of you result.

Cobb-Douglass function

- Charles W. Cobb and Paul H. Douglas (1928) *"A Theory of Production"* AER: cited as one of the top 20 most influential paper in economics
- General form: $Q = K^{\alpha}L^{\beta}$;
  - In our example 6.E, $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$
- Mathematically, the Cobb-Douglass production function is a **homogenous function**.
  - $\alpha + \beta$ is the degree of homogeneity.

<u>Definition:</u> A function is said to be a homogenous function if

$$f(tx_1, tx_2, \dots, tx_n) = t^m f(x_1, x_2, \dots, x_n)$$

where "m" is called the degree of homogeneity.

The Cobb-Douglass function is a homogenous function

- **Proof:** $(tK)^{\alpha}(tL)^{\beta} = t^{\alpha+\beta}K^{\alpha}L^{\beta} = t^{\alpha+\beta}Q$

Example 6.F: Which one is HM function, and to what degree.

- $z = x^2 + y^2$
- $z = x + y^2$
- $z = x^2 y + x^{\frac{8}{3}} y^{\frac{1}{3}}$
- $z = (\alpha K^{1-a} + \beta L^{1-a})^{\frac{1}{1-a}}$

Mathematical properties of HM function of degree "m".

(i) f_i is degree of homogeneity "m-1".

(ii) Euler theorem: $\sum_{i=1}^{N} f_i x_i = mf(x_1, x_2, \ldots, x_N)$

Why do we use HM function in economics?

- For the production, HM reflects the assumption that production technology exhibits return to scale.
- What is the return to scale?
  - If you *proportionately* increase "K" and "L", how much is the increase in "Q", measuring in terms of its proportion to an increase in factor input.

- Degree of homogeneity can tell us:

$$\alpha + \beta = 1: \text{ Constant return to scale}$$

$$\alpha + \beta > 1: \text{ Increasing return to scale}$$

$$\alpha + \beta < 1: \text{ Decreasing return to scale}$$

Elasticity versus Partial elasticity

- $y = f(x)$ ➔ Elasticity $= \frac{\mathrm{d}y}{dx} * \frac{y}{x} = \frac{d\ln(y)}{d\ln(x)}$
- $z = f(w, x)$
  - Partial elasticity of z with respect on w

$$= \frac{\partial z}{\partial w} * \frac{w}{z} = \frac{\partial \ln(z)}{\partial \ln(w)}$$

  - Partial elasticity of z with respect on x

$$= \frac{\partial z}{\partial x} * \frac{x}{z} = \frac{\partial \ln(z)}{\partial \ln(x)}$$

Example 6.F: Consider a demand function with the form: $Q = Ap^{-0.28}m^{-0.34}$, where $p = price\ \ and\ \ m = income$.

Comparative Static Analysis using partial derivative

- Model → solve for reduced-form equations of endogenous variables.
    - Endo as a function of (i) exogenous variables and (ii) parameters
    - We can apply partial derivate to the derived endogenous equilibrium solution.
    - This allows us to see how endogenous variables respond to the changes in exogenous variables and parameters.

Example 6.G: Recall that in a simple macroeconomic model, the solution for equilibrium output can be given by $y^* = \frac{I_0 + G_0}{1-(1-t)b}$, where t is level of income tax and b is marginal propensity to consume. What about the effect of an increase in "b" and an increase in "t" on output?

6.2.2)Total differential

- Total differential in “y” = Total change in “y”
- Recall the concept of single variable differential
    - $dy = f'(x)dx$
    - $f'(x) = marginal\ change\ in\ "y"\ per\ a\ unit\ of\ change\ in\ x.$
    - dx = units of change in “x”.
- How about total differential in multivariate relationship.

Definition: Suppose $y = f(x_1, x_2, x_3, \dots, x_n)$

$$dy = \sum_{i=1}^{N} f_i dx_i$$

Tips: *When you are taking the total differential, you are just taking all the partial derivatives and adding them up.*

Example 6.H: Suppose $(y, r) = \sqrt{\frac{y}{r}}$. Derive total differential of C.

Example 6.J: Consider the equilibrium function of real income $y^* = f(I_0, G_o, t, b) = \frac{I_0 + G_0}{1-(1-t)b}$. Find total differential of y^*.

6.2.3)Total derivative (aka. Chain rule!)

- For an illustrative purpose, consider the consumption function in **example 6.H.** Suppose that *"y" and "r" grows over time.*
- That is, y = g(t) and r = h(t) where t is the numbers of period from now.
- By the composite function, consumption will be *ultimately* determined by "t".
- Then, how do we calculate the derivative of "c" with respect to "t"?
- This requires the concept of total derivative.

  - Total differential in "c" can be given by

$$dC = \frac{\partial c}{\partial y} dy + \frac{\partial c}{\partial r} dr$$

  - Given this assumption, one obtains that

$$dy = g'(t)dt$$

$$dr = h'(t)dt$$

  - Applying the two expressions above, we obtain that

$$dC = \frac{\partial c}{\partial y} g'(t)dt + \frac{\partial c}{\partial r} h'(t)dt$$

  - As a result, total derivative of C with respect to t is given by

$$\frac{dC}{dt} = \frac{\partial c}{\partial y} g'(t) + \frac{\partial c}{\partial r} h'(t)$$

Example 6.K Consider a consumption function with $C = t^2 + \sqrt{y} - r^3$ and $y = 2t^3$ and $r = t + 1$. Derive the total derivative of C with respect to t.

6.2.4)Partial total derivative (aka. Chain rule for multivariate!)

- From the example above, what if both y and r depend on "t" and "s"?

- Consumption would then be ultimately driven by "t" and "s".

- The concept is changed to *partial total derivative*: $\frac{\partial c}{\partial t}$ *and* $\frac{\partial c}{\partial s}$

Following the same method as we applied to total derivative, we know that

$$dC = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial s}ds\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial s}ds\right)$$

As a result,

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial t}\right)$$

Meanwhile, we would yield that

$$\frac{\partial c}{\partial s} = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial s}\right)$$

6.3) The Implicit derivative

- Derivative method applied to an ***implicit function***

Explicit function	**Implicit function**
• $y = 2x - 1$ • $y = x^2 + 2x + 1$ • $y = \ln(2x+1) + 2x + 1$ • $z = \ln(2x+1) + 2y^2x$	• $x^2 + y^2 = 3$ • $xw + yw - w^2 + zx^2 = 9$

For the implicit function, how do we obtain the derivative of y with respect to x?

Method 1: (brute force) → Rewrite y in terms of x.

- Comments:
  - Tedious… and NOT applicable for all the cases.
  - In many cases, the function doesn't admit any closed-form solutions: impossible to rewrite "y" in term of "x".

Example 6.L Suppose that $x^2 + y^2 = 3$, find dy/dx.

Method 2: Apply the implicit function theorem

Theorem: Suppose a given function is written in the form of $F(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof:

Example 6.M Suppose that $\ln(x+2) - y^2 - e^{x+y} = 3,\ Find \frac{dy}{dx}$

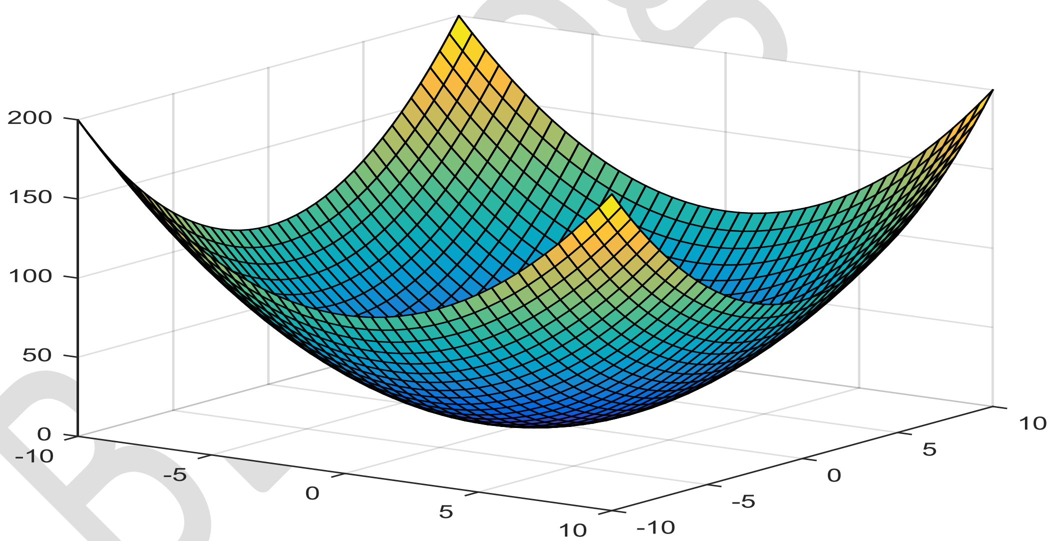
Level set (Contour set)

Definition: Suppose a function $z = f(x, y)$, the level set of "f", denoted by $L(f, z_0)$, is defined as

$$L(f, z_0) = \{(x, y)|\ f(x, y) = z_0\}$$

All possible combinations of (x,y) that generate f(x,y) = z_0.

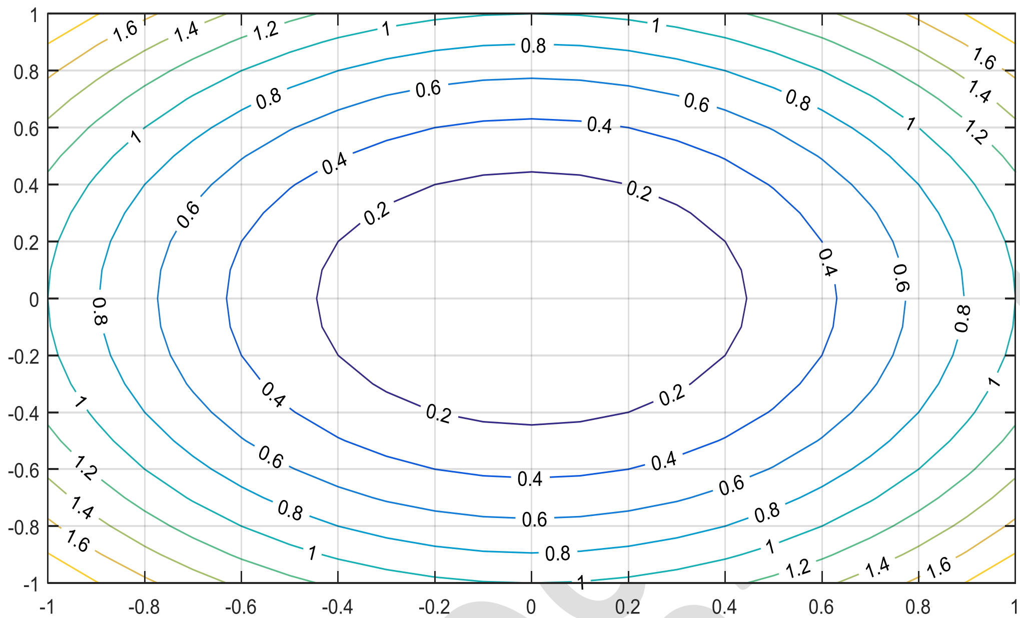
- For example, $z = f(x, y) = x^2 + y^2$



Assuming that $z_0 = 1$, $L(f, 1) = \{(x, y)|x^2 + y^2 = 1\}$

Graphically, $L(f, 1)$ can be represented in a 2-D figure. The shape of the relationship is a circle with unit radius.

Contour of level sets: By varying the value of “z”, we yield several contour of level sets.



Derivative of the level set

- One can apply the second result of the implicit function theorem.

$$\frac{dy}{dx}\Big|_{z=z_0} = -\frac{x}{y}, \qquad i.e.\ \ dz = 0$$

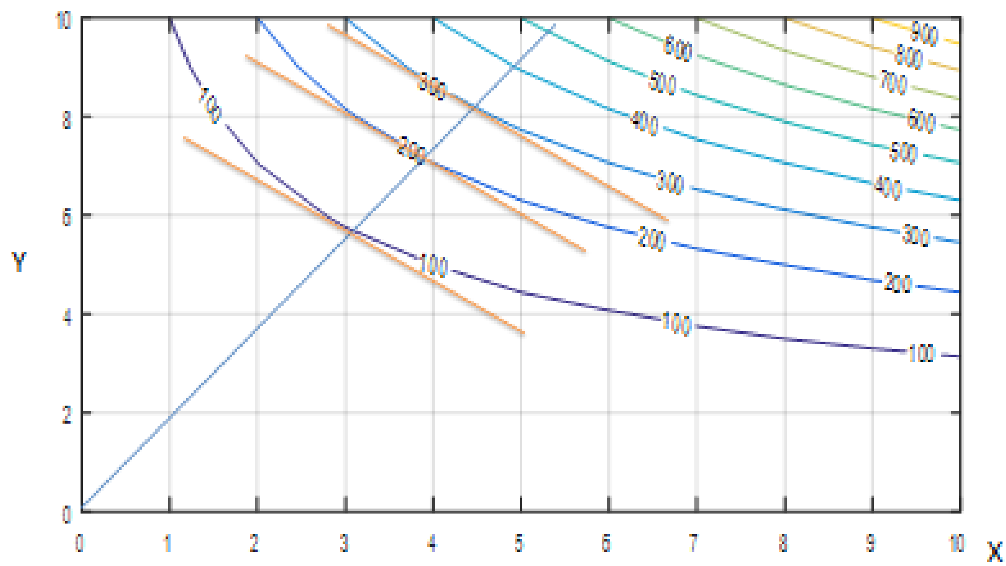
Economic example of the level set

Example 6.O: Suppose that U is the utility function of a consumer. The function takes the following form: $U = U(x, y) = xy^2$ where x is the amount of consumption on good "x" and y is the amount of consumption on good "y".

a) Derive the level set of the utility function. What do we call the level set of utility function in economics?

b) Calculate slope of the level set. What does the slope mean to you in economics?

Locus of indifference curves



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6.3.1) Implicit function and its Application for comparative static analysis

a) Comparative static analysis without explicit solution

Motivating example: Consider a market model where market demand and market supply are given by,

$$Q^d = D(P, y) = a - bP + cy,$$

$$Q^s = S(P) = e + fp \quad S_p > 0$$

Comparative static analysis: ***how does the level of income affect equilibrium output and price?***

Approach for the analysis:

- Mechanically, we solve for the solution (reduced-form solution)
- Apply derivative technique, and check for the sign of derivative.
- Then reach the conclusion over the impact of income (exogenous changes) on equilibrium output and price (the two endogenous variables in our model.)

These are the procedure that we did for the midterm exam. Then, what's new?

Motivating question: would the conclusion be obtained under a wider class of the model equations? (e.g. non-linear function) Could it be shown mathematically? $\frac{\partial Q^*}{\partial y}$, $\frac{\partial P^*}{\partial y}$ =???

Why is this important? → Economics usually want to get some testable implications under more ***robust*** assumptions.

- People might find it less convincing to assume linear demand, and hence arguing that your conclusion is only true under the too restricted assumptions.

- But, people might be more comfortable with the assumption that law of demand and law of supply holds ($D_p < 0$; $S_p < 0$), together with assuming that the production in question is normal goods ($D_y > 0$).

Implicit derivative can be applied into this context. How?

Note first:

$$S(P) = D(P, y)$$

Under some conditions, we should be able to write P in terms of Y as the explicit function, i.e. deriving the reduced-form equation.

b) Comparative static in an aspect of optimization

Consider a profit-maximization problem $\pi(L) = pf(L) - wL$ where $f(L)$ is increasing and concave in L.

Following the first-order condition:

By applying the implicit function, we yield that

c) Comparative static by the Implicit function theorem: *simultaneous approach.*

Consider a simple macroeconomics with three equations

Market clearing condition: $Y = C + I$

Consumption function: $C = f(Y;r) \quad 0 < f_y(Y;r) < 1 \;\; and \;\; f_r(Y;r) \leq 0$

Investment function: $I = g(Y;r,I_0)$

$$0 < g_y(Y;r,I_0) < 1, \;\; g_r(Y;r,I_0) \leq 0 \; and \;\; g_{I_0}(Y;r,I_0) > 0$$

In this model, we are now treating "r" and "I_0" as the two exogenous variables.

By solving for the solution of simultaneous equations, we know that, ***under some regular conditions***, the reduced-form equations must exist,

$$Y^* = Y^*(r, I_0); \qquad C^* = C^*(r, I_0); \qquad I^* = I^*(r, I_0)$$

How can we perform the sensitivity analysis, based on unknown functional form of the reduced-form equation?

First, we know that the three reduced-form equation must satisfy for the three original equations above.

Market clearing condition: $\boldsymbol{Y^*(r, I_0) = C^*(r, I_0) + I^*(r, I_0)}$

Consumption function: $\boldsymbol{C^*(r, I_0) = f(Y^*(r, I_0); r)}$

Investment function: $\boldsymbol{I^*(r, I_0) = g(Y^*(r, I_0); r, I_0)}$

Second, the equilibrium relationship must also hold for the *differential* level.

Chapter 7: Mathematical optimization: **Unconstrained problem**

7.1) General Statement of the problem

Two classes of the optimization problem

(i) *Unconstraint optimization*

$$\max_{x_1,x_2,\ldots,x_N}(\min) f(x_1, x_2, \ldots, x_N)$$

Comments: You are free to choose combinations of "x".

Example:

(ii) Constraint optimization

$$\max_{x_1, x_2, \ldots, x_N} f(x_1, x_2, \ldots, x_N)$$

such that $g(x_1, x_2, \ldots, x_N) = b$

Comments:

a) *M-constraint* (equations) functions

b) Constraint set could be described/represented by ***inequality***,

$x + y = 2$ $v.s.$ $x + y \leq 2$ → Line V.S. Area shape

This topics is the central discussion for EE421.

7.2) Unconstrained optimization

- This section develops toolkits to solve for the solution of the unconstraint optimization problem

7.2.1) Types of optimizer revisited!

Definition: Local optimizer

(x^*, y^*) is a ***local*** optimizer of $f(x, y)$ if and only if, for all (x', y') that are closed to (x^*, y^*),

$f(x^*, y^*) > f(x', y')$ (local max)

$f(x^*, y^*) < f(x', y')$ (local min)

How do mathematicians define the closed-to set?

- x^* ➔ x' in the neighborhood of x^*
  - $(x^* - \varepsilon, x^* + \varepsilon)$
  - Distant between x' and x^* is less than ε.

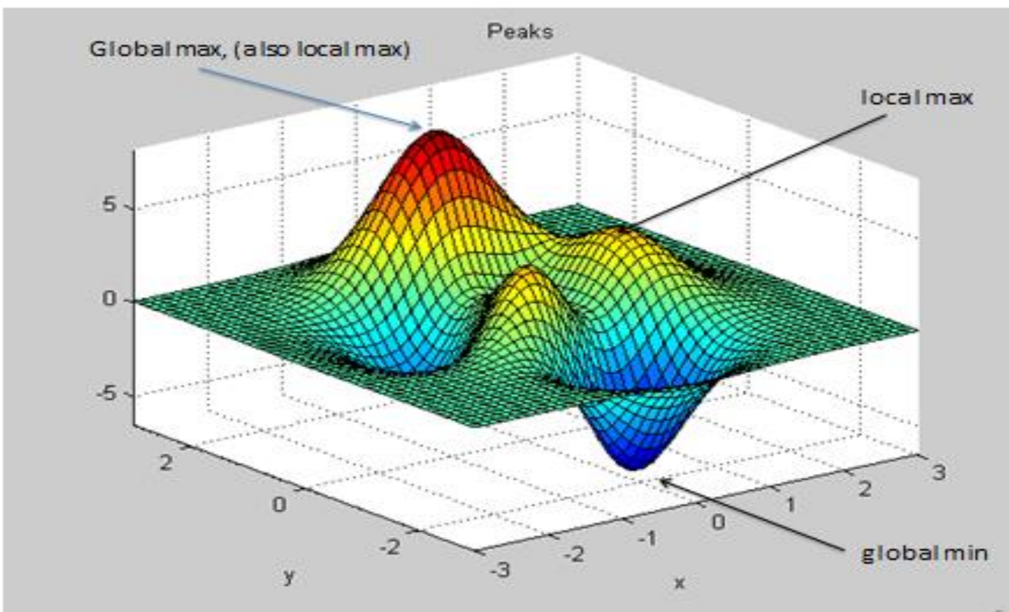
- (x^*, y^*) ➔ (x', y') in the neighborhood of (x^*, y^*)
  - Distant between (x', y') and (x^*, y^*) is less than ε.
  - $\forall (x', y')$: $\sqrt{(x' - x^*)^2 + (y' - y^*)^2} < \varepsilon$

Definition: Global optimizer

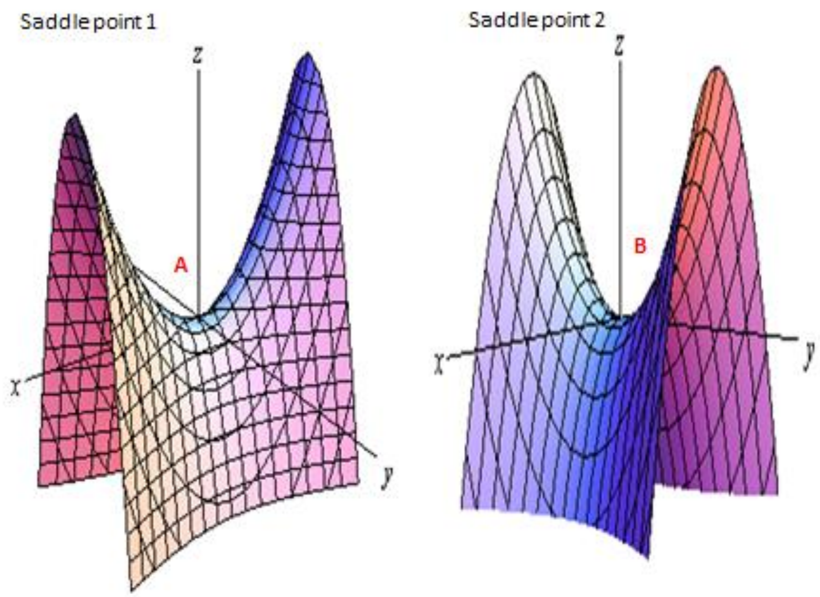
(x^*, y^*) is a ***global*** optimizer of $f(x, y)$ if and only if, for all (x', y') in the **entire domain set** that defines "f",

$f(x^*, y^*) > f(x', y')$ (global max)

$f(x^*, y^*) < f(x', y')$ (global min)



- **Comments**:
  - If the point is local max, it must be local max on both axes (directions).
  - If the point is local max on one axis (direction), but local min on the other axis (direction), the point is neither considered as local max nor local min. Instead, the point is called "***saddle point***".



Definition: Saddle point 1

(x^*, y^*) is a *saddle point* of $f(x, y)$ if and only if, for all (x', y') that are closed to (x^*, y^*),

$f(x^*, y^*) > f(x^*, y')$ (max in y)

$f(x^*, y^*) < f(x', y^*)$ (min in x)

Definition: Saddle point 2

(x^*, y^*) is a *saddle point* of $f(x, y)$ if and only if, for all (x', y') that are closed to (x^*, y^*),

$f(x^*, y^*) < f(x^*, y')$ (min in y)

$f(x^*, y^*) > f(x', y^*)$ (max in x)

7.2.2) Unconstraint (constraint-free) optimization: theorem

- We start out with a 2-varaible case. Later, we will generalize the result into N-variable case.

Theorem: *Properties of Local optimizer*

Suppose that $f(x, y)$ is a twice-differentiable function. That is, both ∇f and Hessian matrix exist. We would observe the following two properties for which the local optimizer (x^*, y^*) must hold.

- $\nabla f\ (x^*, y^*\ ) = 0$. **(First-order necessary condition)**
- If (x^*, y^*) is the local maximizer (minimizer) of $f(x, y)$, then $f(x, y)$ is concave (convex) at (x^*, y^*). **(Second-order necessary condition)**

Theorem: *Necessary and Sufficient condition*

This theorem is commonly known as the two-step cooked procedure for locating local optimizers. Given that f(x,y) is twice differentiable function,

(i) **First-order condition:** Locate (x^*, y^*) such that Gradient of the function is equal to zero.

(ii) **Second-order condition:** Distinguishing the type of local optimizers by checking for the type of curvatures at the point located from (i).

a. Concave → local maximizer
b. Convex → local minimizer

Concavity/convexity of a function

In single variable case, we check for the concavity/convexity of a function using the second-order derivative.

- This is fine, because we only have one axis/direction of change to consider.

For the multivariate case, it is **no longer** appropriate to simply look at the second-order partial derivative. Mathematically, we check for the concavity/convexity of a multivariate function using the **second-order total differential**.

- $f(x\quad,y\quad)$ is said to be *convex* at the point (x^*, y^*) if and only if

$$d^2f(x^*,y^*) > 0$$

- $f(x\quad,y\quad)$ is said to be *concave* at the point (x^*, y^*) if and only if

$$d^2f(x^*,y^*) < 0$$

Example 7.A Derive the expression for the second-order differential of f(x,y)

Theorem:

- If Hessian matrix is *positive definite*, then $d^2 f(x^*, y^*) > 0$ for all possible combinations of dx and dy.
  - Implies that "f" is convex!

- If Hessian matrix is *negative definite*, then $d^2 f(x^*, y^*) < 0$ for all possible combinations of dx and dy.
  - Implies that "f" is concave!

Theorem:

Suppose that A is a ***symmetric*** matrix.

- A is positive definite if the determinants of all the ***leading principal minors*** are positive.
- A is negative definite if the determinants of all the ***leading principal minors*** have alternate signs, *with* the negative value for the determinant of the initial leading principal minor A_1.

Comments: Since "H" is symmetric, we can apply the theorem to check for the definiteness of "H" using the above theorem. An important question remained is the definition of what so called leading principle minors.

Definition: Leading principal minors

- All the sub-matrices of "A", indexed by A_i, each of which is expanded along the diagonal of the original matrix A.

$$A = \begin{bmatrix} 1 & -4 & 3 \\ -4 & 5 & 0 \\ 3 & 0 & 9 \end{bmatrix} \Rightarrow A_1 = [1]; \quad A_2 = \begin{bmatrix} 1 & -4 \\ -4 & 5 \end{bmatrix}; \quad \mathrm{A}_3 = A$$

Note: There exists "N" leading principal minors for an N-by-N matrix.

Example 7.B: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$; $A = \begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}$

Find all the sub-matrices of A and check for definiteness of A.

From the theorems stated above, we can conclude that, *under the 2-variable case*, second-order condition requires that

- For the minimizer, H is positive definite = "f" is convex
  - $\det(H_1) = f_{xx} > 0$
  - $\det(H_2) = f_{xx} * f_{yy} - f_{xy}^2 > 0$

- For the maximizer, H is negative definite = "f" is concave
  - $\det(H_1) = f_{xx} < 0$
  - $\det(H_2) = f_{xx} * f_{yy} - f_{xy}^2 > 0$

Conclusion:

Example 7.C: Determine all the optimizers of

$$f(x, y) = x^3 - y^3 - 9xy$$

Example 7.D: Determine all the optimizers of

$$f(x, y) = x^2 - 2xy + 3y^2 + 2x - 2y$$

Definition: *Globally concave/convex*

A function is said to be globally concave (convex) if the function is concave (convex) for the entire domain set defining the function.

Following the theorem of definiteness on matrix, we can check for the global property of a function.

Lemma:

- Globally concave:
  - Hessian matrix is negative definite for all (x,y).
- Globally convex
  - Hessian matrix is positive definite for all (x,y).

Example: $\boldsymbol{f(x,y) = e^{(x+y)}}$, $\boldsymbol{g(x,y) = -(x^2 + y^2)}$

Why do we care about this?

- Optimized solution in economics attempt at searching for the first-best, i.e. global solution.
- Unfortunately, mathematical optimization only provides toolkits for searching/identifying local optimizers.

Theorem:

If a function is globally convex (concave), local minimizer (maximizer) is warranted to be global minimizer (maximizer).

N-variable case

Objective function: $z = f(x_1, x_2, x_3, \ldots, x_n)$

FOC.

$f_1 = f_2 = f_3 = \cdots = f_n = 0$ ➔ solve for the solution from FOCs.

SOC.

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

- H is negative definite ➔
  - $|H_1| < 0, |H_2| > 0, |H_3| < 0$ ….
  - sign($|H_i|$) = $(-1)^i$ for each "i".
  - Local min
- H is positive definite ➔
  - $|H_1| > 0, |H_2| > 0, |H_3| > 0$ ….
  - sign($|H_i|$) > 0 for *all* "i".
  - Local max

7.2.3) Some economics applications

a) Basic profit maximization and Factor inputs decision

- **Structure of the problem**

  a. Firm owns a technology that requires the use of some types of factor inputs.

  b. Technology can be represented by a production function.

     i. For simplicity, I will assume only two types of the factor inputs, capital (K) and labor (L)

     ii. Capital costs firm "r" per unit of installation, and labor costs firm "w" per unit of labor usage.

  c. Firm sells the product to the market, getting back "p" in return, per unit of output sold.

  d. Question under the problem is that "*how much to utilize each type of factor input to get the highest level of profit?*"

Objective function:

Choice variables:

Optimality conditions: FOC

Second-order condition

- Hessian of profit function is

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} pf_{KK} & pf_{KL} \\ pf_{LK} & pf_{LL} \end{bmatrix} = p\begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

- So, we need $\begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$ to be negative definite because $p > 0$.
  - This condition ensures the solution from FOC as a maximizer.
  - What does it mean in economics for having negative definitive in $\begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$?
  - What could it be implied about the mathematical property of the production function?

Example 7.E Suppose $Q = K^{\frac{1}{2}} + L^{\frac{1}{2}}$ and price is fixed, equal to P. Assume that "w" as wages and "r" as rental.

a) Setting up the profit function.

b) Derive the optimal level of capital installation and labor usage.

c) Check the second-order condition to confirm that your answer is correct.

d) Calculate the level of maximized (optimal) profit.

e) What is the optimal level of capital installation and labor usage when P = 1000, w = 20 and r = 10.

Exercise 7.A

Redo the **example 7.E**, but using the following Cobb-Douglas function: $Q = K^2L^3$ as the production technology of the firm.

- Derive the demand function for both capital and labor.
- Discuss some important properties of demand function of capital and labor.
- Derive the optimal profit function, and discuss some properties of the function.

b) Multi-plant problem

Structure of the problem:

- Firm operates in a market, and sells only a single type of product in the market.

- Firm has several plants that can be used to fulfill the production. (Think about plants that are located in different cities, for example. Or, two plants in the same location, but each differs in their production technique.)

- How to optimality choose the capacity of each plant, given a certain level of the total output that firm attempts to sell to the market.

For simplicity, consider a 2-plan example.

- $C^1 = C^1(Q_1) = cost\ of\ plant\#\ 1,\ when\ firm\ chooses\ to\ produce\ Q_1$

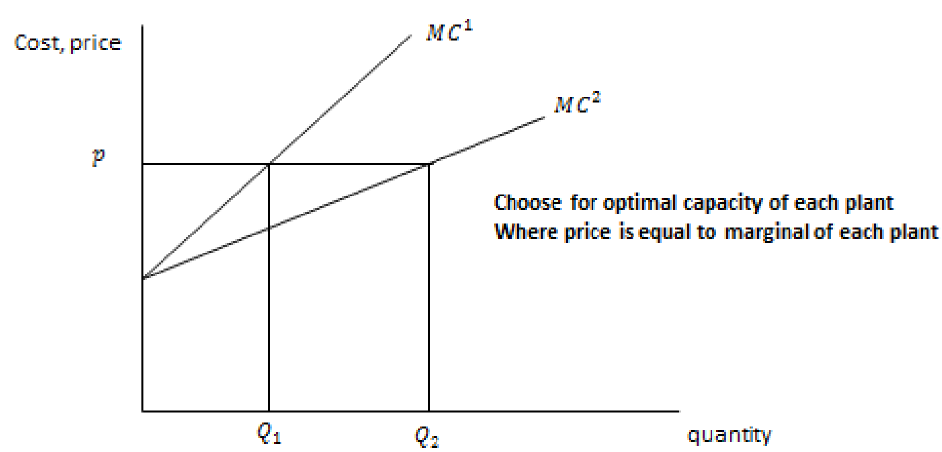
- $C^2 = C^2(Q_2) = cost\ of\ plant\#\ 2,\ when\ firm\ chooses\ to\ produce\ Q_1$

The total production of these two plants must fulfil the total output (Q) that firm is trying to sell to the market. That is, $Q_1 + Q_2 = Q$

First, consider a simple case where firm is a price taker. That is, it treats price (p) as given.

Optimality conditions:

Setting up mathematical problem: Graphical



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Example 7.F

Suppose that P = 25 and the cost function for each plant is given by,

$$C^1(Q_1) = 2Q_1^2 + 5Q_1 + 10$$

$$C^2(Q_2) = 2Q_2^2 + 3Q_2 + 15$$

Q_1^* and Q_2^* that maximizes the profit?

Example 7.G Multi-plant problem of a monopolist.

Suppose a monopolist faces the market demand given by: $P = 10 - Q$

$$TC(Q_1) = 2Q_1^2 + 5Q_1 + 10$$

$$TC(Q_2) = 2Q_2^2 + 3Q_2 + 15$$

Determine $Q_1^* \ and \ Q_2^*$ that maximizes the profit.

c) Multi-product problem

Structure of the problem

- Firm produces different types of products, and distributes those products to different markets.

- Each product can be sold at different prices.

    - In each market, price might be given, or can be set if the firm has market power.

    - Start from the case with given price, first.

Example 7.H:

Suppose the total cost function of a perfectly competitive firm is given by

$$TC(Q_1, Q_2) = 2Q_1^2 + Q_1 Q_2 + Q_2^2.$$

Determine the optimal level of (Q_1, Q_2).

Example 7.I

Suppose $P_1 = 35 - Q_1$ and $P_1 = 33 - Q_2$. Determine the optimal level of output for the two types of product if monopolist's cost function can be given by,

$$C = 2\mathrm{Q}_1^2 + \mathrm{Q}_1\mathrm{Q}_2 + 2\mathrm{Q}_2^2.$$

d) Price discrimination: Third-degree price discrimination

Conventional set-up for the single-product problem a monopolist is given as follows

- Suppose the monopolist faces a downward sloping demand curve, given by: $P = p(Q)$. And the cost function is given by $C(Q)$.
- What is the optimal policy for the monopolist?
  - Choose Q so that monopolist's profit is maximized.
  - Requires setting Q such that: $MR(Q) = MC\,(Q)$.
- Recall that $MR(Q) = P\left(1 - \frac{1}{|\varepsilon_d|}\right)$.
- Given this, we know that:

$$P\left(1 - \frac{1}{|\varepsilon_d|}\right) = MC$$

$$\frac{P}{MC} = \frac{|\varepsilon_d|}{|\varepsilon_d| - 1}$$

- Two implications for pricing setting of the monopolist.

(i) Monopolist will **<u>NOT</u>** produce in the region that demand is inelastic.

$$\frac{P}{MC} = \frac{|\varepsilon_d|}{|\varepsilon_d| - 1}$$

(ii) Having produced under elastic region, $\frac{P}{MC} > 1$.

- Optimal price setting entails firm to set their price above the marginal cost, i.e. setting "mark-up" optimally.
- The optimal mark-up pricing inversely related the elasticity of demand.

- The more elastic demand monopolist faces, the lower degree of mark-up.

Comments:

- Simple, single pricing.
    - In the industrial organization, this is called "single *linear-pricing*".
    - Pricing is linear in the sense that price-per-unit is set the same across different consumers.
    - While it's true that the more you buy, the more you pay in totals. However, each unit of your purchase, price-per-unit remains the same.
- **Is there any pricing strategy that could improve the outcome?**
    - Yes, by the price discrimination strategy.

What is the price discrimination?

- Any pricing strategy aimed at further increasing the profit, i.e. to higher level of profit than that could be attained under single linear-pricing method.
- Might be possible because buyers in the market are actually *heterogeneous*.
    - Rich/poor… High-demand/ low-demand… variety of choices/limited choices
- So, pricing scheme that is based on some socio-economic conditions might be helpful. (That's why it is called "discriminatory".)

First-degree price discrimination

- The strongest version of the price discrimination strategy.
- Varying price for every single units of the purchase.

Second-degree price discrimination

Third-degree price discrimination

- The two versions of the price discrimination discussed above require great deal of information, and therefore it is costly for the practical implementation.
- The third-degree price discrimination proposes a comprised solution in which it is easier to implement.
- That is, monopolist charges for N linear pricing with respect to N different types of consumers. (multiples linear pricing)
- While there is a continuum of buyers in the market, it's possible that we can group those buyers into N groups, each has shared some similar characteristics in economics aspect.

Structure of the problem:

- A single monopolist firm selling its product to N groups of buyers; for simplicity, let's assume two: namely A and B.

  Group A: $\boldsymbol{P_A = D_A(Q_A)}$

  Group B: $\boldsymbol{P_B = D_B(Q_B)}$

  - Note: each group MUST differ in their characteristics and some socio-economic profiles. → to be explained later!
- The product costs the same across different types of consumers.
  - It's not price discrimination if the product costs differently across types of consumers. (You don't know what in fact causes the price to be high in one, but low in the other market. It could be attributed to cost differentials.)

  Cost: $\boldsymbol{C = C(Q) = C(Q_A + Q_B)}$

Example

Consider a plastic producer that produces plastic and sells its product to two different types of users. The first type uses the plastic in dental clinics. The second type uses the plastic as inputs for manufacturing some other products. Let's call the first group as "D"; meanwhile, the second group as "M"

$$P_D = 300 - Q_D$$

$$P_M = 100 - \frac{1}{2} Q_M$$

Suppose further that marginal cost is "$2" per a unit of production.

Find the profit-maximizing output under the third-degree price discrimination.

e) Cournot model

- So far, we have discussed the only two *extreme* cases of market structure under which a representative firm operates: *perfect competitive and monopolist*.
- In many industries, the market has more than one firm, but not too many. Firm's behavior, and hence equilibrium outcome, in those markets are different from the theoretical predictions that we obtain from the model built upon those two extreme assumptions.
- Micro-theorists have developed some models that can be used to understand and predict the behavior of firms under the situation when there are few numbers of firms operating.

- This is known as the ***Cournot model***.

- What the Cournot model captures:

  - Prediction of equilibrium outcome of market in which there are few numbers of firms.

    - Few enough that each firm has to make their decision based on some strategic interactions.

    - A situation is called *strategic interaction* if the outcome/pay-off that a person would receive depends upon the decision/action taken by others, or vice versa.
      - Chess game
      - B&S: battle of the sex

  - What is the equilibrium outcome?
    - Optimality
    - Market clearing or self-sustaining action.

Example: The simple Duopoly market, Optimal behavior

Suppose the 2 firms in the market: namely firm A and firm B. Each firm faces the same market demand: $P = 50 - 2Q$ where Q is total output in the market, P is price per unit. What is the optimal action for firm A when the cost function is given by $C = 2Q_A$?

Definition: Best response function

Let (a_1, a_2) be action profiles that players can choose in a game. Define $u^1(a_1, a_2)$ *and* $u^2(a_1, a_2)$ as the pay-off that arise from having the two players chosen the profile (a_1, a_2). The best response function of each player, indexed by i, is the contingent plan that prescribes the decision rule of its own, based on all possible actions that could be chosen by the other players.

$$BR^1 = a_1^* \in \max_{a_1} u(a_1, a_2) ==> a_1 = BR^1(a_2)$$

$$BR^2 = a_2^* \in \max_{a_2} u(a_1, a_2) ==> a_2 = BR^2(a_1)$$

Definition: Nash-equilibrium

(a_1^N, a_2^N) is said to be a Nash equilibrium if

(i) $u^1(a_1^N, a_2^N) \geq u^1\left(a_1 , a_2^N\right)$
(ii) $u^2(a_1^N, a_2^N) \geq u^2\left(a_1^N, a_2 \right)$

Result: Nash equilibrium profile solves for the solution of simultaneous equations given by best responses functions.

Equilibrium prediction under Cournot

Merger / Collusion

7.3 Constrained optimization

Statement of the problem: Comparison

Constrain-free optimization	**Constraint optimization**
$\max\ (\min)_{x,y}\ f(x,y)$	$\max\ (\min)_{x,y}\ f(x,y)$ such that $g(x,y) = c$

3

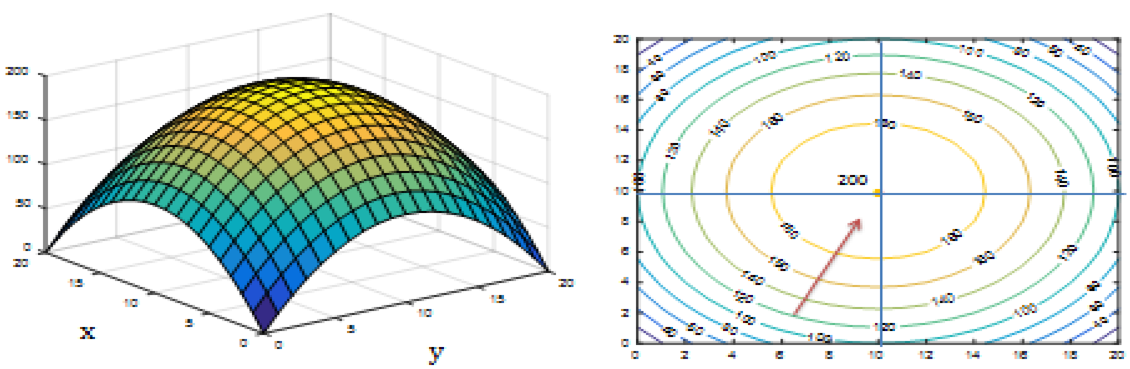
Statement of the problem: Comparison

Constrain-free optimization	**Constraint optimization**
$\max\ 200-(x-10)^2-(y-10)^2$	$\max\ 200-(x-10)^2-(y-10)^2$ such that $x+y=10$

4

3-D curve of the objective function

2D representation of the objective function

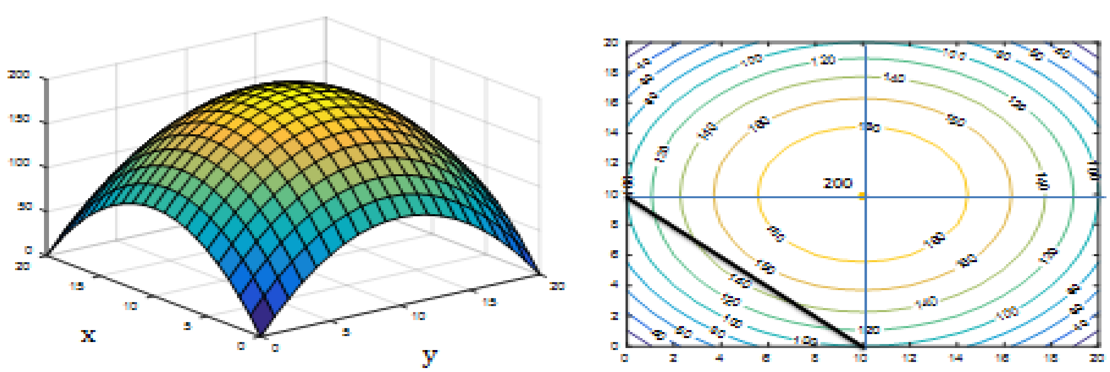


X= 10 and y = 10 is the optimum point under constraint-free optimization problem.

Introducing a constraint set would affect the possible choice set.

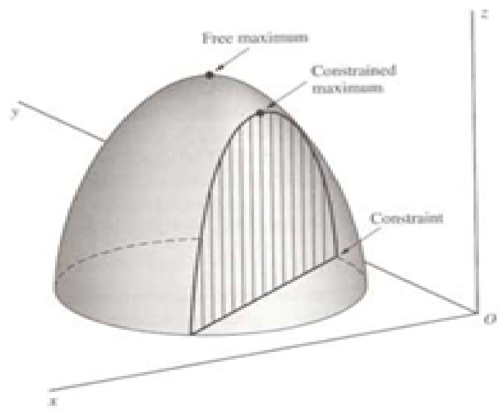
3-D curve of the objective function

2D representation of the objective function



Can we still choose x = 10 and y = 10 when the constraint is imposed?

The Impact of introducing constraint



- Lower the optimum value of the objective function.

- Our goal is to come up with solution methods that help finding the constrained optimum point.

7

Solution methods

1) Substitution method

Example:

$$\max_{x,y} \quad x \quad y$$

subject to $x + y = 7$

Limitations:

- Functional form of the constraint function is too complicate.
  - Ex. $\ln\left(\frac{x}{y}\right) + x^2 - \frac{y^2}{x^2+1} = \frac{3}{xy} + 2$
- More than single constraint set.
  - In general case, there can be more than one constraint.
  - Doable, but difficult/impractical to reduce the problem into unconstrained optimization problem.

2) LaGrange method

Construct a new function called the **LaGrange function**.

The function takes the following form:

$$L(x, y, \lambda; c) = f(x, y) + \lambda[\boldsymbol{c} - \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})]$$

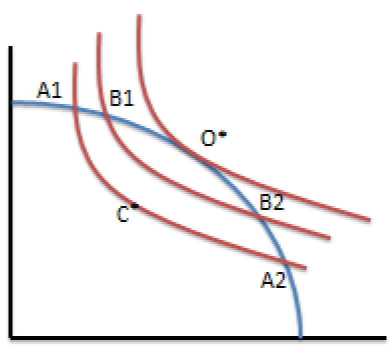
λ is the LaGrange multiplier.

(an auxiliary variable introduced for computational purpose, to be explained later.)

- Under some regularity conditions, the solution to the constrained optimization is the stationary point for the LaGrange function.
  - A stationary point of a function is the point where "first-order derivatives" are equal to zero.
- For the LaGrange function, they are (x^*, y^*, λ^*) such that

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$$

A geometric interpretation



- Keep moving the objective function until the function is tangent to the constraint set.

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Example: Solve for the problem $\max_{x,y} x \quad y \quad s.t. \;\; c_0 x + c_1 y = c_2$

The Interpretation of LaGrange multiplier

Definition: Optimal value function

$f^*(\mathbf{c}) = f(x^*(\boldsymbol{c}), y^*(\boldsymbol{c}))$ as the optimal value of the object function.

Envelop theorem: Given the optimal value function,

$$\frac{d\mathrm{f}^*(\boldsymbol{c})}{dc_i} = \frac{dL(x, y; \boldsymbol{c})}{dc_i}$$

Proof:

Following the theorem, one knows that $\boldsymbol{\lambda^*(c) = \frac{df^*(c)}{dc_2}}$

By approximation, we know that:

$$f^*(\boldsymbol{c} + dc_2) - f^*(\boldsymbol{c}) \approx \lambda^*(\boldsymbol{c})dc_2$$

If we relax the constraint set/value (c_2) imposed on our optimization by a small amount of dc_2 units, on its *first approximation*, the optimized value of the objective function would be changed by $\lambda^*(\boldsymbol{c})\, dc_2$.

The second-order condition: confirming the result

- Similar to the unconstrained optimization problem, we need to confirm our result.
- If the objective function is <u>concave</u> at the stationary point (along the constraint set), solution to the first-order condition is then confirmed to be a constrained maximizer.
- To do this, we check for the property of the second-order derivative matrix of the LaGrange function.
- By checking the property of the so called "*Bordered*" Hessian matrix.

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{bmatrix}$$

Example: Check the concavity and convexity of the example above.

Checking conditions:

- For maximum, determinant of the bordered Hessian must be POSITIVE.
- For minimum, determinant of the bordered Hessian must be NEGATIVE.

Wrap's up: 2-step cooked procedure.

- Step 1: Forming the LaGrange function
- Step 2: Derive stationary points of the LaGrange function (Solving the FOCs of LaGrange function)
- Step 3: Checking concavity/convexity of function along the constraint set.
  - Convex: Determinant of Bordered Hessian is NEGATIVE. (Minimum)
  - Concave: Determinant of Bordered Hessian is POSITIVE. (Maximum)

Some economics examples:

Example: Regulated v.s. Unregulated monopolist

Suppose that a monopolist profit function is given by

$$\pi(x, y) = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

where "x" is the level of output sold to the first type of consumer and "y" is the level of output sold to the second type of consumer.

a) Find the optimal level of output that generate the highest profit

b) What happen if the government limits the total of quantity output to be equal to 79 units.

c) If the government allows for the production limit to be 80 units, what would happen to the optimized level of profit?

Example: **Utility maximization problem**

Suppose that a household has the preference relationship defined by the utility function $U(x,y) = 2x^{0.6}y^{0.3}$. Suppose further that the price of goods x is P_x and the price of goods y is P_y. Given M as the budget of this household, find the optimal bundle of consumption for goods x and goods y.

Exercise: Redo the example with the new utility function $U(x, y) = (x-1)^{\frac{1}{2}}(y-4)^{\frac{1}{2}}$.

The obtained solutions, $x^*(p_x, p_y, M)$ *and* $y^*(p_x, p_y, M)$, from the above utility maximization problem (UMP) yield us the so called "*Marshallian demand function*". The function is HM degree zero in prices and income.

Indirect utility function:

The optimized level of utility can be derived by plugging the optimal consumption bundle into the utility function. Economically, the optimized utility function, expressed in terms of prices and income, is called the ***indirect utility functio***n. Mathematically, the indirect utility function is the optimal value function attained from the optimization problem.

$$v(p_x, p_y, M) = U\left(\boldsymbol{x}^*(\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{M}), \boldsymbol{y}^*(\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{M})\right)$$

Properties:

(i) HM degree of zero in prices and income.

Proof:

(ii) $\boldsymbol{x}^* = -\frac{\frac{\partial v}{\partial p_x}}{\frac{\partial v}{\partial p_M}}$ and $\boldsymbol{y}^* = -\frac{\frac{\partial v}{\partial p_y}}{\frac{\partial v}{\partial p_M}}$ (Roy's identity)

Proof:

Compensated demand function (Hicksian demand curve)

If you studied EE311, you had seen this concept before, the Hicksian demand curve.

An alternative form of the demand function can be derived from the *optimal bundle that minimizes the level of expenditure.* Hicks introduced this concept after Marhsall's introduction on demand function, and it has become known since then under the name of the *compensated demand function*.

Hicks's idea is that, instead of searching for the bundle that maximize the utility under budge, household is then assumed to choose the bundle that minimizes the level of expenditure needed for acquiring a certain level of utility. The problem under Hicks's supposition is then

$$\boldsymbol{mix\ \ p_x x + p_y y\ \ \ s.t.\, u(x, y) = \bar{u}}$$

Solution to the above Hicks's problem can be derived in a similar way to the problem stated out by Marshall. That, is we use the LaGrange method.

The resulting optimization problem yields us the solution where

$$\boldsymbol{x^h = x^h(p_x, p_y, \bar{u})} \text{ and } \boldsymbol{y^h = y^h(p_x, p_y, \bar{u})}$$

Given the Hicksian demand bundle, the optimized level of expenditure, commonly called the expenditure function, is given by,

$$\boldsymbol{e(p_x, p_y, \bar{u}) = p_x x^h(p_x, p_y, \bar{u}) + p_y y^h(p_x, p_y, \bar{u})}$$

Duality in Household decision problem

(i) $x^h\left(p_x, p_y, v(p_x, p_y, M)\right) = x^*(p_x, p_y, M)$

(ii) $x^*\left(p_x, p_y, e(p_x, p_y, \bar{u})\right) = x^h(p_x, p_y, \bar{u})$

(iii) Roy's Identity:

$$x^* = -\frac{\frac{\partial v}{\partial p_x}}{\frac{\partial v}{\partial p_M}} \text{ and } y^* = -\frac{\frac{\partial v}{\partial p_y}}{\frac{\partial v}{\partial p_M}}$$

(iv) Expenditure function:

$$x^h = \frac{\partial e(p_x, p_y, \bar{u})}{\partial p_x} \text{ and } y^h = \frac{\partial e(p_x, p_y, \bar{u})}{\partial p_y}$$

Example: **Cost function**

Suppose that firm has the production technology given by $Q = \sqrt{K}+\sqrt{L}$. To acquire each unit of capital (K) and labor (L), firm must pay for the unit cost of r and w, respectively.

a) Derive the optimal combinations of factor inputs.

b) What is the cost function?

Chapter 8 Integration and its application

8.1) Definition and Basic integration formula

- An **antiderivative** of a function *f* is a function *F* such that $F'(x) = f(x)$

  In differential notation, $dF = f(x)dx$

- Integration states that

  $\int f(x)dx = F(x) + C$ if only $F'(x) = f(x)$

- Basic Integration

  Properties:

1.	$\int k\,dx = kx + C$ k is a constant
2.	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $n \neq -1$
3.	$\int x^{-1}\,dx = \int \frac{1}{x}dx = \int \frac{dx}{x} = \ln x + C$ for $x > 0$
4.	$\int e^x\,dx = e^x + C$
5.	$\int kf(x)\,dx = k\int f(x)\,dx$ k is a constant
6.	$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example: Find $\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right)dx$

Example: If y is a function of x such that $dy/dx = 8x - 4$ and $y(2) = 5$, find y.

Example: In the manufacture of a product, fixed costs per week are \$4000. (Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period.) If the marginal-cost function is

$$\frac{dc}{dq} = 0.000001\left(0.002q^2 - 25q\right) + 0.2$$

where c is the total cost (in dollars) of producing q pounds of product per week, find the cost of producing 10,000 lb in 1 week.

Integration by substitution method

Example: a. $\int (x+1)^{20} dx$ b. $\int 3x^2 (x^3+7)^3 dx$

8.2) Definite integral

- If *f* is continuous on the interval [*a, b*] and *F* is any antiderivative of *f* on [*a, b*], then

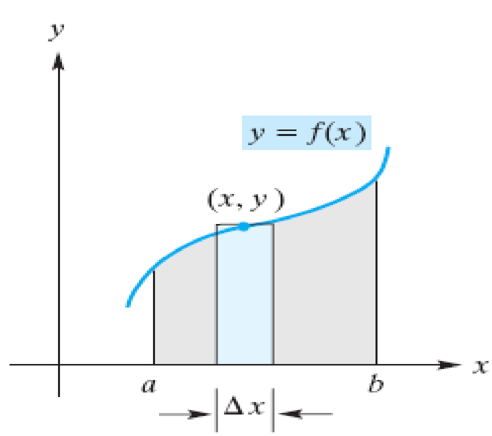
$$\int_a^b f(x)dx = F(b) - F(a)$$

- Property of definite integral

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,; \quad c \in [a, b]$$

- Definite integral as the area under curve
  - Suppose we are interested in finding the area under f(x) bounded by (over the interval) [a,b].
  - Geometrically.
    - The width of the vertical element is Δx. The height is the *y*-value of the curve.
  - The area is defined as

$$\sum f(x)\Delta x \to \int_a^b f(x)dx = area$$



Example: Find the area of the region bounded by the curve $y = 6 - x - x^2$ for $x \in [-3,2]$

Example: Find the area of the region under the curve $y = x^3$ for $x \in [-2,2]$

8.3) Applications of integration in economics.

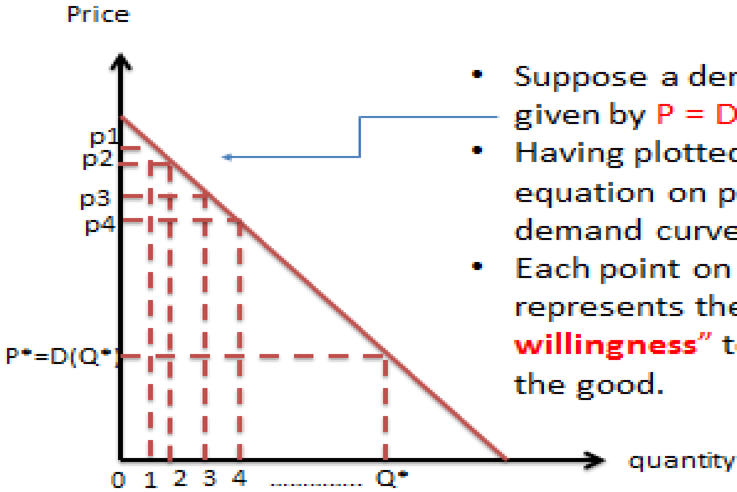
Recoverability: getting back to the level function from the marginal function

- $MU(x) \to U(x)$
- $MC(Q) \to C(Q)$
- $I(t) \to K(t)$

Welfare analysis:

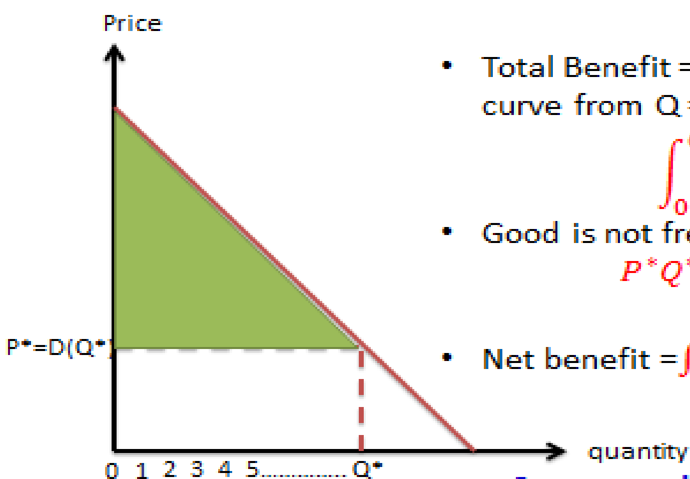
- We normally want to measure the desirability of market outcome or market allocation.
- There are several welfare measurements
- Consumers' and producers' surplus (CS/PS) are the most popularized concept that most economists use.
  - Given information about demand and supply curve, we can measure CS and PS by applying the (definite) integration as the tool for measurement.
- What are they?

Consumer surplus



- Suppose a demand function is given by P = D(Q).
- Having plotted the demand equation on p-q axis, we get a demand curve.
- Each point on the demand curve represents the "**maximum willingness**" to pay for each unit of the good.

Consumer surplus



- Total Benefit = Area under the demand curve from Q = 0 upto Q = Q*

$$\int_0^{Q^*} D(Q)dQ$$

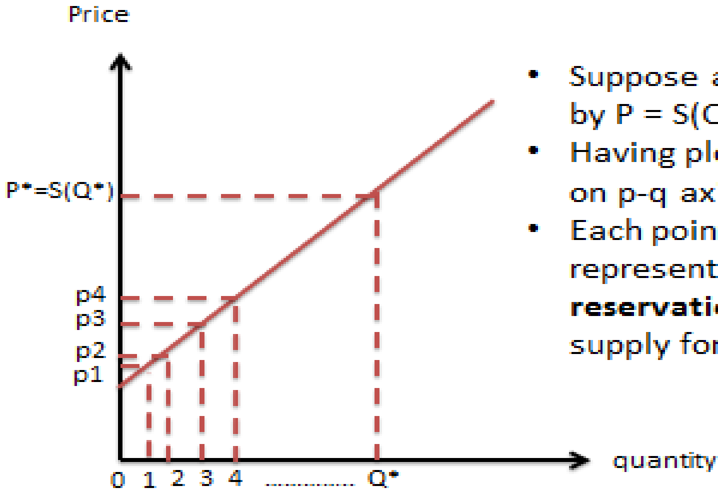
- Good is not free!

$$P^*Q^* = D(Q^*)Q^*$$

- Net benefit = $\int_0^{Q^*} D(Q)dQ - D(Q^*)Q^*$

Consumers' surplus =
Area under demand, but above price

Producer surplus



- Suppose a supply function is given by P = S(Q).
- Having plotted the supply equation on p-q axis, we get a supply curve.
- Each point on the supply curve represents the "**minimum reservation price**" required to supply for each unit of the good.

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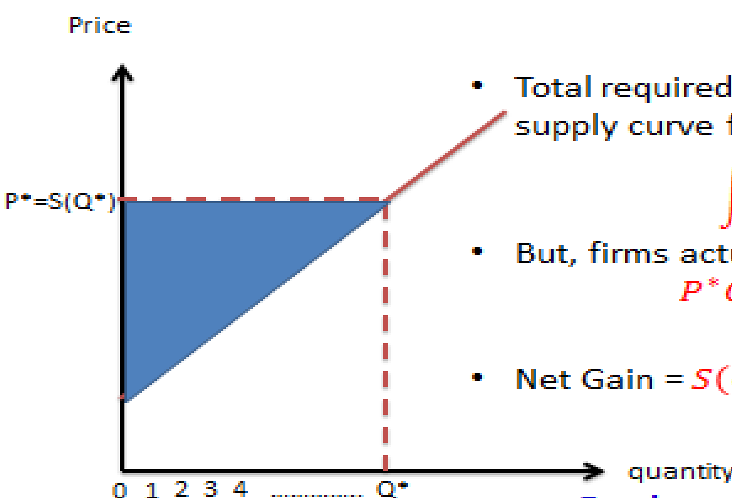
Producer surplus



- Total required revenue = Area under the supply curve from Q = 0 upto Q = Q*

$$\int_0^{Q^*} S(Q)dQ$$

- But, firms actually get

$$P^*Q^* = S(Q^*)Q^*$$

- Net Gain = $S(Q^*)Q^* - \int_0^{Q^*} S(Q)dQ$

Producers' surplus = area above supply, but under price

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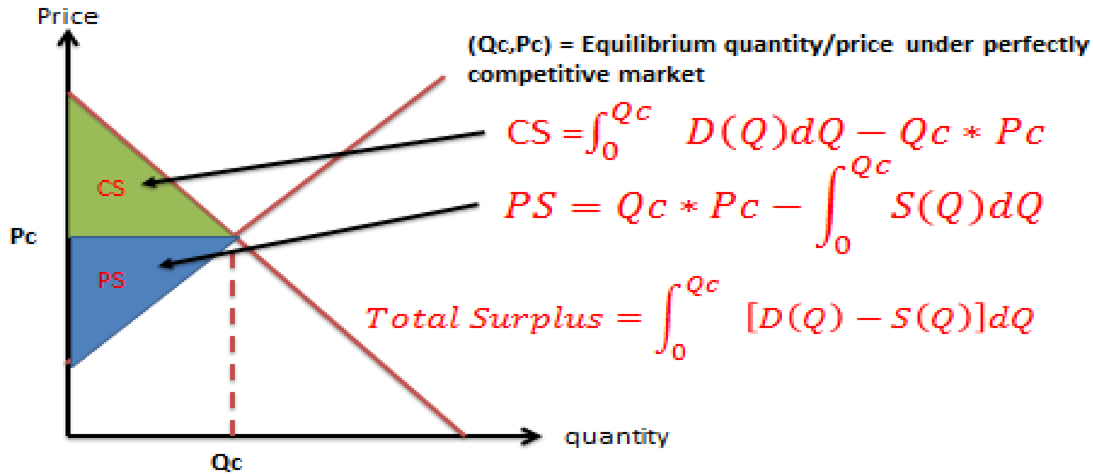
Welfare under perfectly competitive equilibrium



Example: Suppose the market demand equation is $Q = \frac{1}{2}(25 - P)$ and the market supply equation is $Q = \frac{1}{2}(P - 1)$. Evaluate the welfare at the market equilibrium price and quantity.

First-best equilibrium:

- Economist typically refers to the "perfectly competitive" equilibrium/allocation as the benchmark for the first-best equilibrium.
  - Proof should be given in Micro311, for the part that discusses about General Equilibrium.
  - Rigorous proof can be found in Debreu (1959) "*Theory of value*".
- In what sense?
  - In the sense that the allocation generates the biggest total surplus of the two (consumers/producers) combined.

- Any market inventions on the perfectly competitive market would do more harm than good.
  - Any interventions, such as taxes/subsidies/regulations, would do harm, rather than doing good, e.g. yielding us lowered welfare

- Any markets with the structure deviating from the assumptions imposed under perfectly competitive market would yield lower welfare.

Example: Suppose that the market demand is given by P = 30 – Q and the cost function is given by $C(Q) = \frac{1}{2}Q^2$. Calculate the Deadweight loss that arises under monopoly market.

8.6) Basic differential equation (optional)

Suppose that $y = f(t)$, but don't know the function form of "f".

However, we know that **"y"** satisfies the equation given by a function

$$G(y, y', y'', y''', \ldots, y^n, t) = 0$$

where $y^n = \frac{d^n y}{dt^n}$ → n-th derivative.

The problem that we attempt to uncover the unknown form of "f" by using the information obtained in "G" is mathematically called the **differential equation problem!**

For example:

$G(y', y'', y''', \ldots, y^n) = y' + 2y - 3 + 4t$

- first-order differential equation (linear)

$G(y', y'', y''', \ldots, y^n) = y' + 2y - 3$

- first-order differential equation (linear/autonomous case)

$G(y', y'', y''', \ldots, y^n) = y' + 2y$

- first-order differential equation (linear/autonomous case/ homogenous case)

$G(y', y'', y''', \ldots, y^n) = y'' + 3y' - \frac{1}{5}y$

- second-order differential equation (linear/autonomous case/homogenous case)

$G(y', y'', y''', \ldots, y^n) = (y'')(y') + 3y' - \frac{1}{5}y$

- second-order differential equation (non-linear system; second-degree)

General form: **The n-order linear differential equation**

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_2 y'' + a_1 y' + a_0 y + k + bt = 0$$

- If k = 0 → Homogenous case
- If b = 0 → Autonomous case
- n = 1 → firs-order equation

In this lecture, we provide some basic tools that can be used to solve for the solution of **first-order homogenous linear differential equation**.

$$a_1 y' = a_0 y + k + bt$$

Case 1: Solution to Homogenous and autonomous equation ($k = 0, b = 0$)

Case 2: Solution to non-homogenous autonomous equation ($b = 0$)

The solution takes the form: $y = y_h + y_s$

where

(i) y_h = Solution to the homogenous equation (when k = 0)

(ii) $y_s =$ Solution to the steady-state equation (what is it?)

Stability of the solution:

The solution is said to be stable if $\lim_{t \to \infty} y(t) \to y_s \; (steady\ solution)$

What condition do we need?