Linear Transformations

Review of Linear Transformations

A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T and $\mathbb{R}^m$ is called the **codomain** of T.

- Ex 1. Suppose we have two vectors, v_1 and v_2 and we would like to map them to the vectors $u_1 = v_1 + v_2$ and $u_2 = v_1 - v_2$. This can be done by building the matrix A, such that $Av = u$, such that

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- Ex 2. Given a matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \text{and } \mathbf{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

  Then this is a transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$. We can say a transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by $T(\mathbf{x}) = A\mathbf{x}$.

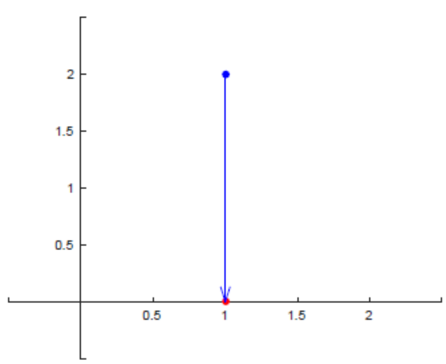
- Ex 3. Given a matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

  We have

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

  So this transformation projects points in $\mathbb{R}^2$ to the x-axis. See the picture below. If we have $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .Then $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a projection of $\mathbf{x}$ at x-axis.



Definition of linear transformations

A transformation T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Notice: Every matrix transformation is a linear transformation and satisfies (i) and (ii).

- Ex 4. Given a matrix A and two vectors $\mathbf{u}, \mathbf{v}$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{and } \mathbf{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

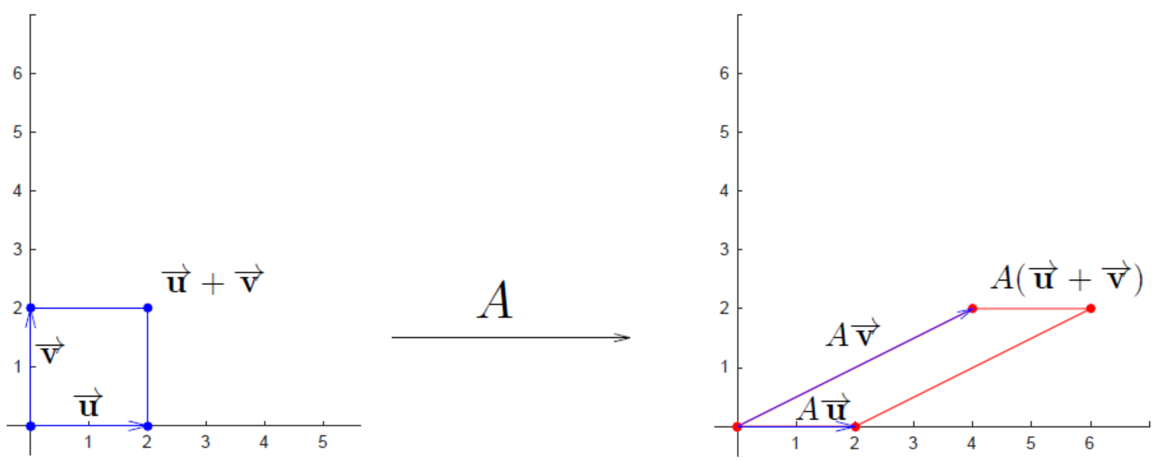
Then we have

$$A\mathbf{u} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{and } A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Then

$$A(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = A\mathbf{u} + A\mathbf{v},$$

which means that this transformation is a linear transformation. We also can see this linear transformation below.



This transformation deforms the squre as if the top of the square were pushed to the right while base is held fixed. So it called **shear transformation**.

- Ex 5.

  Given a matrix A and two vectors $\mathbf{u}, \mathbf{v}$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{and } \mathbf{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

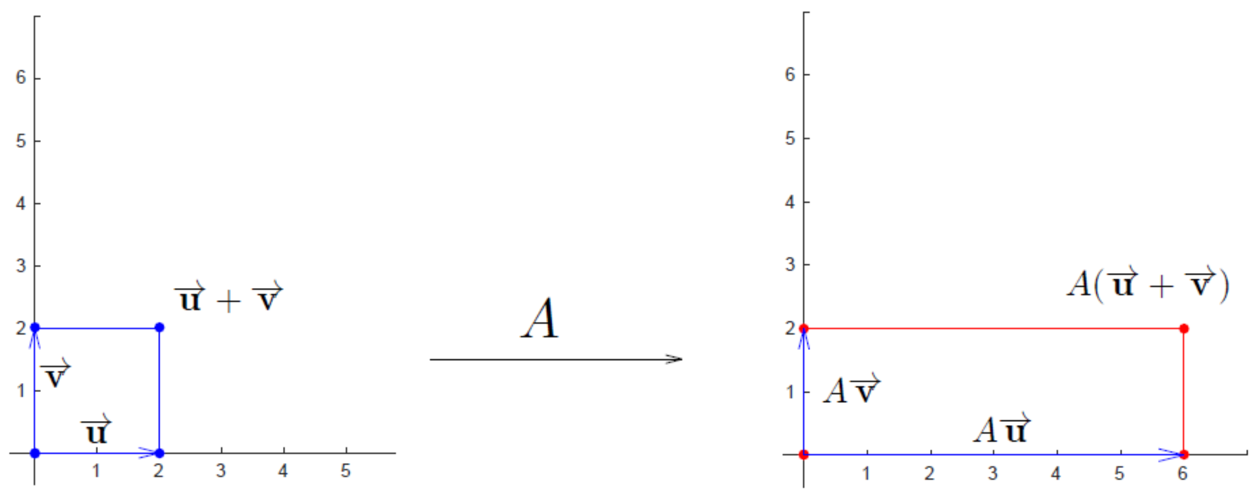
Then we have

$$A\mathbf{u} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \text{and } A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Then

$$A(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = A\mathbf{u} + A\mathbf{v},$$

which means that this transformation is a linear transformation. We also can see this linear transformation by pictures:



Linear Transformations in MATLAB

Our previous examples focused on lines and are pretty intuitive. In fact, they could be done by hand with a pen and paper. However, this applies to more complicated objects too. Suppose we have a set of coordinates, X, and we want to transform it linearly in some fashion with a matrix A.

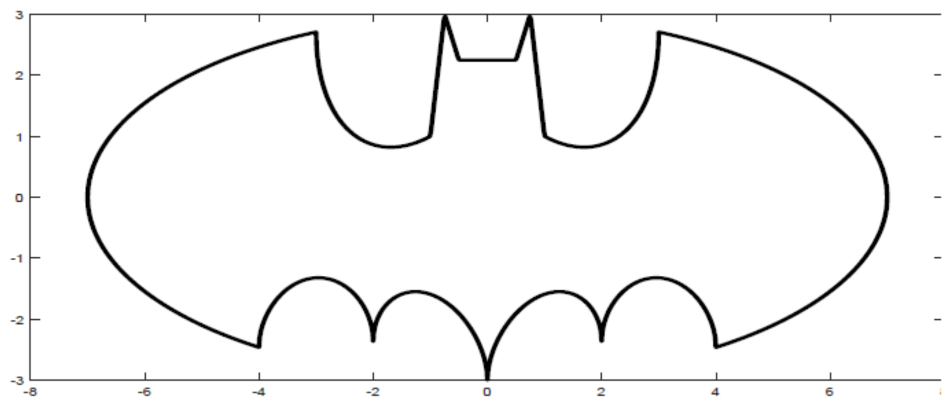
For fun, let's import the Batman logo into MATLAB. Download the datasets 'batmanx.txt' and 'batmany.txt' and put them into your MATLAB folder. Then, in the command window, type:

```
>> X = [csvread('batmanx.txt');csvread('batmany.txt')];
```

Now, to see what it looks like, type

```
>> plot(X(1,:),X(2,:),'k','LineWidth',3);
```

You should see something like this pop up:



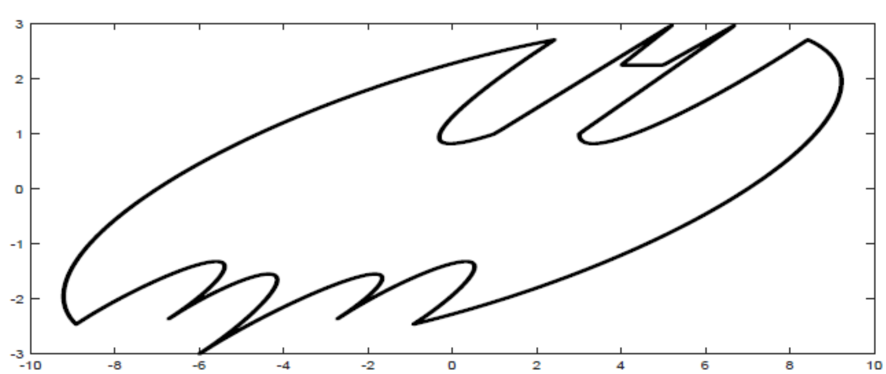
Right, now let's do a shear transformation like in Ex 4. Set A as it is there, by typing:

```
>> A = [1,2;0,1];
```

Right, now let's do our transformation, store it as Y, and plot it. Type

```
>> Y = A*X; plot(Y(1,:),Y(2,:),'k','LineWidth',3);
```

You should see something like:



which has sheared the object like before with the vectors. You can experiment with other transformations with this data, just modify the transformation by modifying A and repeating the steps.

Some Transformation Matrices

You should know how to perform the most used transformations, such as rotations, reflections, scaling, shearing, and projects. These transformation matrices are given below as a quick reference. However, it is not comprehensive. You may need to modify or test different transformations to get the one you want.

- **Rotation**: To rotate by angle θ counterclockwise, set your transformation matrix A as

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- **Reflection**: Reflection against the x-axis , set your transformation matrix A as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Reflection**: Reflection against the y-axis, set your transformation matrix A as

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Reflection**: Reflection against the line $y = x$, set your transformation matrix A as

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- **Reflection**: Reflection against the line $y = -x$, set your transformation matrix A as

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

- **Reflection**: Reflection against the origin, set your transformation matrix Aas

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Scaling**: To scale by length a in all directions, set your transformation matrix A as

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

- **Shear**: To shear by length b in the horizontal direction, set matrix A to

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$