Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\,\forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

Let $S_t = W_t - C_t$

$R_t = R_f + w^*_t (R_M - R_f)$

$u(c_t, t) = \delta^t \frac{C_t^{(1-\gamma)}}{1-\gamma}$

$B(W_T, T) = \delta^t \frac{W_T^{(1-\gamma)}}{1-\gamma}$

$U_c = E_{T-1}(B_W R_{T-1})$

$\delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}\left[\delta^T W_T^{-\gamma} R_{T-1}\right]$

$= E_{T-1}\left[\delta^T (S_{T-1} R_{T-1})^{-\gamma} R_{T-1}\right]$

$= \delta^T E_{T-1}\left[R_{T-1}^{1-\gamma}\right] (W_{T-1} - C_{T-1})^{-\gamma}$

Rearrange: $C_{T-1} = \frac{(\delta E_{T-1}(R_{T-1}^{1-\gamma}))^{-\frac{1}{\gamma}}}{1 + (\delta E_{T-1}(R_{T-1}^{1-\gamma}))^{-\frac{1}{\gamma}}} W_{T-1}$

$= \frac{a_1}{1+a_1} W_{T-1}$

Let $a_1 = (\delta E_{T-1}(R_{T-1}^{1-\gamma}))^{-\frac{1}{\gamma}}$

$= (\delta E_T (R_{T-1}^{1-\gamma}))^{-\frac{1}{\gamma}}$

$E_{T-1}(B_W R_{T-1}] = R_f E_{T-1}[B_W)$

$E_{T-1}(R_{T-1}^{-\gamma} R_{r,T-1}) = R_{f,T-1} E_{T-1}(R_{T-1}^{-\gamma})$

$\therefore$ Weights is independent to level of wealth and consumption

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

Let $a_1 = \left(\delta E_{T-1}[R_{T-1}^{1-\gamma}]\right)^{-\frac{1}{\gamma}}$

$= \left(\delta E_T[R_{T-1}^{1-\gamma}]\right)^{-\frac{1}{\gamma}}$

$$J(W_{T-1}, T-1) = \delta^{T-1}\left(\frac{C_{T-1}^{(1-\gamma)}}{1-\gamma}\right) + \delta^T E_{T-1}\left[\frac{(R_{T-1}(W_{T-1}-C_{T-1}))^{(1-\gamma)}}{1-\gamma}\right]$$

$$= \delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^{(1-\gamma)} \frac{W_{T-1}}{1-\gamma} + \delta^T E_{T-1}\left[\frac{R_{T-1}^{(1-\gamma)}\left(W_{T-1}-\frac{a_1}{1+a_1}W_{T-1}\right)^{(1-\gamma)}}{1-\gamma}\right]$$

$$= \delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^{1-\gamma}\frac{W_{T-1}}{1-\gamma} + \delta^T E_{T-1}\left[R_{T-1}^{(1-\gamma)} \frac{W_{T-1}^{(1-\gamma)}}{(1-\gamma)(1+a_1)^{(1-\gamma)}}\right]$$

$$= \delta^{T-1}\frac{W_{T-1}^{1-\gamma}}{(1-\gamma)(1+a_1)^{1-\gamma}}\left(a_1^{1-\gamma} + \delta E_{T-1}[R_{T-1}^{(1-\gamma)}]\right)$$

Let $b_1 = \left(a^{1-\gamma} + \delta E(R_{T-1}^{(1-\gamma)})\right)/(1+a)^{1-\gamma} = [a_1^{1-\gamma} + a_1^{-\gamma})/(1+a_1)^{1-\gamma}$

$= (a_1 a_1^{-\gamma} + a_1^{-\gamma})/(1+a_1)^{1-\gamma}$

$= \frac{a_1^{-\gamma}(a_1+1)}{(1+a_1)^{1-\gamma}}$

$= \frac{(1+a_1)^{\gamma}}{a_1^{\gamma}}$

$= \left(\frac{a_1}{1+a_1}\right)^{-\gamma}$

$a_1 = \left(\delta E_T(R_{T-1}^{1-\gamma})\right)^{-\frac{1}{\gamma}}$

$a_1^{-\gamma} = \delta E_T(R_{T-1}^{1-\gamma})$

Let $b_1 = \left(\frac{a_1}{1+a_1}\right)^{-\gamma}$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$v_c(C_{T-2}, T-2) = E_{T-2}\left[J_W(W_{T-1}, T-1)\ R_{T-2}\right]$$

$$C_{T-2}^{-\gamma} = \delta^{T-1} E_{T-2}\left[b_1 W_{T-1}^{-\gamma}\ R_{T-2}\right]$$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}\left[b_1 (S_{T-2} R_{T-2})^{-\gamma} R_{T-2}\right]$$

$$= \delta b_1 E_{T-2}(R_{T-2}^{1-\gamma})(W_{T-2} - C_{T-2})^{-\gamma}$$

$$C^*_{T-2} = \frac{(\delta b_1 E_{T-2}[R_{T-2}^{-\gamma}])^{-\frac{1}{\gamma}}}{1+(\delta b_1 E_{T-1}(R_{T-2}^{1-\gamma}])^{-\frac{1}{\gamma}}} W_{T-2}$$

$$= \frac{a_2}{1+a_2} W_{T-2}$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = U(C^{*}_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)]$$

$$= \delta^{T-2} C^{1-\gamma} / 1-\gamma + E_{T-1}[\delta^{T-1} b_1 W_{T-1}^{1-T} / 1-\gamma]$$

$$= \delta^{T-2} \left(\frac{a_2}{1+a_1}\right)^{1-\gamma} W_{T-2}^{1-\gamma} / 1-\gamma + \delta^{T-1} E_{T-1}\left[b_1 R_{T-2}^{1-\gamma} \frac{W_{T-2}^{1-\gamma}}{(1+a_1)^{1-\gamma}}\right]$$

$$= \delta^{T-2} \frac{W_{T-2}^{1-\gamma}}{(1-\gamma)(1+2\gamma)} \left(a_2^{1-\gamma} + \delta E_{T-2}(b_1 R^{*(1-\gamma)}_{T-2})\right)$$

$$= \delta^{T-1} b_2 W_{T-2}^{1-\gamma} / 1-\gamma$$

where $\left[a_2^{(1-\gamma)} + \delta E(R^{*\gamma}_{T-2})\right) / (1+a_2)^{(-\gamma)}$