

## CHAPTER 4

### Basic Matrix Algebra and Applications

#### Topics: Economic Applications of Matrix Algebra

##### Outline:

How to use matrix to solve system of simultaneous equations

Partial market equilibrium

The effect of specific tax on market equilibrium

Simple Macroeconomic Model

IS – LM

## How to use matrix to solve system of simultaneous equations

When we have the system of simultaneous equations:

- 1.) Identify endogenous variables, exogenous variables, parameters:

This is a system of  $m$  linear equations with  $n$  endogenous variables:

$$\begin{array}{c}
 \left. \begin{array}{l}
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2 \\
 \vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m
 \end{array} \right\} m \text{ eqs}
 \end{array}$$

$n$  Vars

- 2.) Reduce the system of equations to have number of equations equal to number of endogenous variables and to have only necessary information

That is, we need to have:

$$\text{number of equations} = \text{number of endogenous variables}$$

- 3.) Rearrange every equations in the system so that all endogenous variables are on one side and all other exogenous variables on the other side

From:

$$\begin{array}{l}
 ax_1 = k - bx_2 \\
 dx_2 = h - cx_1
 \end{array}$$

To:

$$\begin{array}{l}
 ax_1 + bx_2 = k \\
 cx_1 + dx_2 = h
 \end{array}$$

- 4.) Rewrite the system into matrix form:

$$\begin{array}{l}
 ax_1 + bx_2 = k \\
 cx_1 + dx_2 = h
 \end{array}$$

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} k \\ h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} k \\ h \end{bmatrix}_{2 \times 1}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{n \times n} x_{n \times 1} = d_{n \times 1}$$

$A$  is matrix of parameters that are in front of endogenous variables.

$x$  is column vector of endogenous variables to solve for its final values as a function of exogenous variables and parameters

$d$  is column vector of parameters that don't multiply with endogenous variables and exogenous variables

5.) Check that  $\det A$  must not equal to 0, so that  $A^{-1}$  exists.

$$|A| \neq 0$$

,where  $\det A$  can be computed by:

- If a square matrix  $A$  is a 2 by 2 matrix, then  $\det A$  is equal to:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

- If a square matrix  $A$  is a 3 by 3 matrix, then  $\det A$  is equal to:

$$|A|$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

- $n^{\text{th}}$ -order determinant by Laplace Expansion

Let  $A$  be a matrix in  $R^{n \times n}$ . The determinant of  $A$  or  $\det A$  is given by  $|A|$ .

$$|A| = \begin{cases} \sum_{j=1}^n a_{ij} C_{ij} & i = 1, 2, \dots, n \text{ (expansion by } i^{\text{th}} \text{ row)} \\ \sum_{i=1}^n a_{ij} C_{ij} & j = 1, 2, \dots, n \text{ (expansion by } j^{\text{th}} \text{ column)} \end{cases}$$

We call  $C_{ij}$ , Cofactor of element  $a_{ij}$ .

$$C_{ij} = (-1)^{i+j}M_{ij}$$

The minor of element  $a_{ij}$ ,  $M_{ij}$ , is the determinant of the matrix resulting from deleting row  $i$  and column  $j$  of matrix  $A_{n \times n}$ . In other words, a "minor" is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix.

Note: If determinant is found by using Cofactor for elements from the WRONG row or column, the determinant will be zero!!!!

$$\begin{vmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \quad |C_{21}| = - \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \quad |C_{22}| = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 \quad |C_{23}| = - \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$|A| = 4(-3) + 1(10) + 2(1) = 0$$

6.) Use inverse matrix **or** Cramer's rule to find the solution of the system of simultaneous equations:

6.1) Inverse matrix:

$$\begin{aligned} Ax &= d \\ A^{-1}Ax &= A^{-1}d \\ Ix &= A^{-1}d \end{aligned}$$

$$x = A^{-1}d$$

,where we can find  $A^{-1}$  by

$$A^{-1} = \frac{1}{|A|} \cdot adjA$$

$$adjA = [C_{ij}]^T$$

6.2) Cramer's rule:

$$x_k = \frac{|A_k|}{|A|}$$

$|A_k|$  is the determinant of matrix  $A$  when replacing column  $k$  of matrix  $A$  by vector  $d$ .

**Partial market equilibrium**

**Demand:**

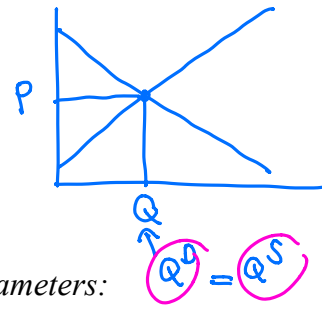
$$Q^D = a - bP$$

**Supply:**

$$Q^S = -c + dP$$

**Equilibrium:**

$$Q^D = Q^S$$



1.) Identify endogenous variables, exogenous variables, parameters:

Endogenous variables:  $Q^D, Q^S, P$

2.) Reduce the system of equations to have number of equations equal to number of endogenous variables

How many equations will we be working with? 3

3.) Rearrange every equations in the system so that all endogenous variables are on one side and all other exogenous variables on the other side

**Demand:**

$$Q^D = a - bP \Rightarrow$$

$$Q^D + bP = a$$

$$Q^D + 0Q^S + bP = a$$

**Supply:**

$$Q^S = -c + dP$$

**Equilibrium:**

$$Q^D = Q^S$$

**Demand:**

$$1 Q^D + 0 Q^S + b P = a$$

**Supply:**

$$0 Q^D + 1 Q^S - d P = -c$$

**Equilibrium:**

$$1 Q^D - 1 Q^S + 0 P = 0$$

4.) Rewrite the system into matrix form:

coeffi in Demand<sup>1</sup>: row 1 →

coeffi in Supply<sup>2</sup>: row 2 →

equi can: row 3 →

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q^D \\ Q^S \\ P \end{bmatrix} = \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$$

col 1 coeffi of  $Q^D$

col 2 coeffi of  $Q^S$

col 3 coeffi of  $P$

Column vector

$$1Q^D + 0Q^S + bP$$

$$Ax = d$$

$$A^{-1}Ax = A^{-1}d$$

$$Ix = A^{-1}d$$

$$x = A^{-1}d$$

5.) Check that  $\det A$  must not equal to 0, so that  $A^{-1}$  exists.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$$

$$A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}_{3 \times 3}$$

$$\det A = |A| = + (1 \times 1 \times 0) + (0 \times -d \times 1) + (b \times 0 \times -1) - (1 \times 1 \times b) - (-1 \times -d \times 1) - (0 \times 0 \times 0)$$

$$\det A = |A| = -b - d \neq 0$$

if  $|A| \neq 0$ ,  $A$  is a non singular  
 $A$  has  $A^{-1}$ .

6.) Use inverse matrix **or** Cramer's rule to find the solution of the system of simultaneous equations:

6.1) Inverse matrix:

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}A$$

$$\text{adj}A = [C_{ij}]^T$$

$$C = \begin{bmatrix} (-1)^{1+1} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{1+2} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{1+3} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} \\ (-1)^{2+1} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{2+2} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{2+3} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} \\ (-1)^{3+1} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{3+2} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} & (-1)^{3+3} \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} (-1)^{1+1} \begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix} & (-1)^{1+2} \begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix} & (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1} \begin{vmatrix} 0 & b \\ -1 & 0 \end{vmatrix} & (-1)^{2+2} \begin{vmatrix} 1 & b \\ 1 & 0 \end{vmatrix} & (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1} \begin{vmatrix} 0 & b \\ 1 & -d \end{vmatrix} & (-1)^{3+2} \begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix} & (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} (-1)^{1+1}(-d) & (-1)^{1+2}d & (-1)^{1+3}(-1) \\ (-1)^{2+1}b & (-1)^{2+2}(-b) & (-1)^{2+3}(-1) \\ (-1)^{3+1}(-b) & (-1)^{3+2}(-d) & (-1)^{3+3}(1) \end{bmatrix}$$

$$C = \begin{bmatrix} -d & -d & -1 \\ -b & -b & 1 \\ -b & d & 1 \end{bmatrix}$$

$$\text{adj}A = C^T = \begin{bmatrix} -d & -b & -b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \text{adj}A = \frac{1}{-b-d} \begin{bmatrix} -d & -b & -b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix}$$

Therefore,

$$x = A^{-1}d$$

$$\begin{bmatrix} Q^D \\ Q^S \\ P \end{bmatrix} = \frac{1}{-b-d} \begin{bmatrix} -d & -b & -b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} Q^D \\ Q^S \\ P \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{b+d} \\ \frac{ad-bc}{b+d} \\ \frac{a+c}{b+d} \end{bmatrix}$$

6.2) Use Cramer's rule:  $x_k = \frac{|A_k|}{|A|}$

$$\begin{bmatrix} Q^D \\ Q^S \\ P \end{bmatrix} = \begin{bmatrix} \frac{|A_1|}{|A|} \\ \frac{|A_2|}{|A|} \\ \frac{|A_3|}{|A|} \end{bmatrix}$$

$$Q^D = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 0 & b \\ 1 & -d \end{vmatrix}}{-b-d}$$

$$Q^S = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix}}{-b-d}$$

$$P = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{-b-d}$$

☺ If we assume  $Q^D = Q^S = Q$ ,

From the model:

**Demand:**  $Q^D = a - bP$

**Supply:**  $Q^S = -c + dP$

**Equilibrium:**  $Q^D = Q^S$

If we assume  $Q^D = Q^S = Q$ , then we can reduce the system of simultaneous equations down to:

**Demand:**  $Q = a - bP$

**Supply:**  $Q = -c + dP$

Rearrange & Rewrite into matrix form:

**Demand:**  $Q + bP = a$

**Supply:**  $Q - dP = -c$

$$\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} a \\ -c \end{bmatrix}$$

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \left| \begin{array}{cc} a & b \\ -c & d \end{array} \right| \\ -b - d \\ \left| \begin{array}{cc} 1 & a \\ 1 & -c \end{array} \right| \\ -b - d \end{bmatrix}$$

☺ If we collect specific tax from supplier  $t$  baht per unit

From the model:

**Demand:**  $Q^D = a - bP^D$

**Supply:**  $Q^S = -c + dP^S$

**Equilibrium:**  $Q^D = Q^S$

$$P^S = P^D - t$$

If we assume  $Q^D = Q^S = Q$  and  $P^D = P$ , then we can reduce the system of simultaneous equations down to:

**Demand:**  $Q = a - bP$

**Supply:**  $Q = -c + dP$  \_\_\_\_\_

Rearrange & Rewrite into matrix form:

**Demand:**  $Q + bP = a$

**Supply:**  $Q - dP = -c$  \_\_\_\_\_

$$\begin{bmatrix} \quad \\ \quad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \left| \begin{array}{cc} a & b \\ -c & d \end{array} \right| \\ -b - d \\ \left| \begin{array}{cc} 1 & a \\ 1 & -c \end{array} \right| \\ -b - d \end{bmatrix}$$

Exercise:

1.)

$$Q^D = 24 - 4P$$

$$Q^S = 13P - 27 \quad (P^* = 3, Q^* = 12)$$

$$Q^D = Q^S$$

2.) Consider a simple model in which two commodities are related to each other

Good 1

Good 2

$$Q_{d1} = Q_{s1}$$

$$Q_{d2} = Q_{s2}$$

$$Q_{d1} = a_0 + a_1P_1 + a_2P_2$$

$$Q_{d2} = c_0 + c_1P_1 + c_2P_2$$

$$Q_{s1} = b_0 + b_1P_1 + b_2P_2$$

$$Q_{s2} = d_0 + d_1P_1 + d_2P_2$$

Find the equilibrium prices and quantities in good 1 and good 2

 **Simple Macroeconomic Model**

Consider simple Keynesian crossing model:

(1.)  $Y = C + I + G$

(2.)  $C = a + bY_d$

(3.)  $Y_d = Y - T$

(4.)  $I = I_0$

(5.)  $G = G_0$

*Identify endogenous variables, exogenous variables, parameters:*

Endogenous variables:.....

*Reduce the system of equations to have number of equations equal to number of endogenous variables*

How many equations will we be working with? .....

$$Y = C + I_0 + G_0$$

$$C = a + bY^d$$

$$Y^d = Y - T$$

Rearrange every equations in the system so that all endogenous variables are on one side and all other exogenous variables on the other side

$$\text{---}Y \text{---}C \text{---}Y^d = \text{---}$$

$$\text{---}Y \text{---}C \text{---}Y^d = \text{---}$$

$$\text{---}Y \text{---}C \text{---}Y^d = \text{---}$$

Rewrite the system into matrix form:

$$\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}$$

$$Y^* = \frac{\begin{vmatrix} \text{---} & -1 & 0 \\ \text{---} & 1 & -b \\ \text{---} & 0 & 1 \end{vmatrix}}{1-b}$$

$$C^* = \frac{\begin{vmatrix} 1 & \text{---} & 0 \\ 0 & \text{---} & -b \\ -1 & \text{---} & 1 \end{vmatrix}}{1-b}$$

$$Y^{D*} = \frac{\begin{vmatrix} 1 & -1 & I_0 + G_0 \\ 0 & 1 & a \\ -1 & 0 & -T \end{vmatrix}}{1-b}$$

Note: We can also work with the system with two equations:

From

$$Y = C + I_0 + G_0$$

$$C = a + bY^d$$

$$Y^d = Y - T$$

To

$$Y = C + I_0 + G_0$$

$$C = a + bY - bT$$

### IS – LM

The Good Market:

$$Y = C + I + G$$

$$C = C_0 + b(1 - t)Y$$

$$I = I_0 - er$$

$$G = G_0$$

The Money Market:

$$M_d = fY - \beta r$$

$$M_s = M_0$$

$$M_d = M_s$$

*Identify endogenous variables, exogenous variables, parameters:*

Endogenous variables:  $Y, C, I, \dots$

*Reduce the system of equations to have number of equations equal to number of endogenous variables*

How many equations will we be working with? .....

$$Y = C + I + G_0$$

$$C = C_0 + b(1 - t)Y$$

$$I = I_0 - er$$

$$M_0 = fY - \beta r$$

*Rearrange*

$$\begin{aligned} Y - C - I + 0r &= G_0 \\ -b(1 - t)Y + C + 0I + 0r &= C_0 \\ 0Y + 0C + I + er &= I_0 \\ fY + 0C + 0I - \beta r &= M_0 \end{aligned}$$

*Rewrite the system into matrix form:*



