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Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

$$2, 3, 4, 6$$

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E[mX_a] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

Ans :

$$P_s = \pi_s m_s 1 \quad \text{and} \quad m_s = \frac{\delta U'(C_s^*)}{U'(C_0^*)} = \delta \frac{C_0^*}{C_s^*}$$

Therefore :

$$p_s = \pi_s \delta \frac{C_0^*}{C_s^*}$$

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3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E\left[m_{01} R_s\right]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\widetilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

Ans

Since:

$$m_{01} = \frac{\delta U'(C_1)}{U'(C_0)} = \delta\left(\frac{C_1}{C_0}\right)^{\gamma-1}$$

FOC:

$$\pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} R_s = 1 \quad \text{— (1)}$$

Then, differentiate (1) :

$$\pi_s \delta (\gamma-1) \left(\frac{C_s}{C_0}\right)^{\gamma-2} R_s \, d\left(\frac{C_s}{C_0}\right) + \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s = 0$$

Rearrange :

$$(\gamma-1)\left(\frac{C_s}{C_0}\right)^{-1} R_s \, d\left(\frac{C_s}{C_0}\right) = -dR_s$$

$$\frac{R_s}{(C_s/C_0)} \, d\left(C_s/C_0\right) = \frac{-dR_s}{(\gamma-1)}$$

$$\frac{R_S}{(C_s/C_0)} \frac{d(C_s/C_0)}{d R_S} = -\frac{1}{\gamma - 1}$$

$$\frac{R_S}{(C_s/C_0)} \frac{d(C_s/C_0)}{d R_S} = \frac{1}{(1-\gamma)}$$ #

4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

$p = \frac{1}{R_f}$

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

Let $p = \begin{bmatrix} 1/1.05 \\ 6 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}$

Then, $\begin{bmatrix} p_1 & p_2 \end{bmatrix} = p' X^{-1}$

$$= \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2476 & 0.7048 \end{bmatrix}$$ #

For the risk neutral probabilities :

$\hat{\pi}_1 = p_1 R_f = 0.2476 \times 1.05 = 0.26$

$\hat{\pi}_2 = p_2 R_f = 0.7048 \times 1.05 = 0.74$

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b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$.

What is the stochastic discount factor for the two states?

Ans : $m_1 = \frac{P_1}{\pi_1} = \frac{0.2476}{0.5} = 0.4952 \approx 0.495$ #

$m_2 = \frac{P_2}{\pi_2} = \frac{0.7048}{0.5} = 1.4096 \approx 1.41$ #

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma \quad \rightarrow \quad \gamma = \frac{E[R_i] - R_f}{\beta_i}$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

a) Ans : $0 = E[m(R_i - R_f)]$

$= E[(a + bR_m)(R_i - R_f)]$

$= aE[R_i] - aR_f + bE[R_m R_i] - bR_f E[R_m]$

$= a(E[R_i] - R_f) + b(E[R_m]E[R_i] + \text{cov}[R_m, R_i] - R_f E[R_m])$

$= (E[R_i] - R_f)(a + bE[R_m]) + b\,\text{cov}[R_m, R_i]$

so, $$E[R_i] - R_f = \frac{-b\,\mathrm{cov}[R_m, R_i]}{a + bE[R_m]}$$

$$= -\frac{\mathrm{cov}[R_m, R_i]}{\sigma_m^2} \frac{b\sigma_m^2}{a + bE[R_m]}$$

$$= -\beta_i \frac{b\sigma_m^2}{a + bE[R_m]}$$

Therefore, $$\gamma = -\frac{b\sigma_m^2}{a + bE[R_m]} \quad \#$$

b) Ans:

Risk free asset: $$\frac{1}{R_f} = E[a + bR_m]$$

$$a = \frac{1}{R_f} - bE[R_m]$$

Market portfolio:

$$1 = E[(a + bR_m)R_m]$$

$$= aE[R_m] + bE[R_m^2]$$

$$1 = aE[R_m] + b(\sigma_m^2 + E[R_m]^2) \quad \text{---} (1)$$

Substitute a into (1)

$$1 = \left[\frac{1}{R_f} - b\cancel{E[R_m]}\right]E[R_m] + b(\sigma_m^2 + \cancel{E[R_m]^2})$$

$$1 = \frac{E[R_m]}{R_f} + b\sigma_m^2$$

Find b :

$$1 - \frac{E[R_m]}{R_f} = b\sigma_m^2$$

$$\frac{R_f - E[R_m]}{R_f} = b\sigma_m^2$$

$$\frac{R_f - E[R_m]}{R_f\sigma_m^2} = b$$ #

Find a :

$$a = \frac{1}{R_f} - bE[R_m]$$

$$a = \frac{1 \times \sigma_m^2}{R_f \times \sigma_m^2} - \left[\frac{R_f - E[R_m]}{R_f\sigma_m^2}\right]E[R_m]$$

$$= \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f\sigma_m^2}$$ #

c) Ans :

$$a + bE[R_m] = \frac{\sigma_m^2 + E[\cancel{R_m}](E[\cancel{R_m}] - \cancel{R_f})}{R_f\sigma_m^2} + E[\cancel{R_m}]\left[\frac{\cancel{R_f} - E[\cancel{R_m}]}{R_f\sigma_m^2}\right]$$

$$= \frac{\cancel{\sigma_m^2}}{R_f\cancel{\sigma_m^2}} = \frac{1}{R_f}$$

Therefore :

$$\gamma = -\frac{b\sigma_m^2}{a + bE[R_m]}$$

$$= -\left(\frac{R_f - E[R_m]}{\cancel{R_f}\,\cancel{\sigma_m^2}}\right)\cancel{\sigma_m^2}\,\cancel{R_f}$$

$$= E[R_m] - R_f$$ #