Exercise IV

1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

4 Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It may be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for H. There is more than one answer.

5 Let $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$. It may be verified that $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

6 Suppose $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$. Explain why $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

Decide whether each statement is true or false, and give a reason for each answer. Here V is a nonzero finite-dimensional vector space.

7 If $\dim V = p$ and if S is a linearly dependent subset of V, then S contains more than p vectors.

8 If S spans V and if T is a subset of V that contains more vectors than S, then T is linearly dependent.

Solution Part 1

1. Let $A = [\mathbf{v}_1 \quad \mathbf{v}_2]$. Row operations show that

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

Not every row of A contains a pivot position. So the columns of A do not span $\mathbb{R}^3$, by Theorem 2 in Section 2.2. Hence $\{\mathbf{v}_1, \mathbf{v}_2\}$ is not a basis for $\mathbb{R}^3$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not in $\mathbb{R}^2$, they cannot possibly be a basis for $\mathbb{R}^2$. However, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are obviously linearly independent, they are a basis for a subspace of $\mathbb{R}^3$, namely, Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.

2. Set up a matrix A whose column space is the space spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, and then row reduce A to find its pivot columns.

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns of A are the pivot columns and hence form a basis of Col $A = W$. Hence $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for W. Note that the reduced echelon form of A is not needed in order to locate the pivot columns.

3. Neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ is in H, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ cannot be a basis for H. In fact, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for the *plane* of all vectors of the form $(c_1, c_2, 0)$, but H is only a *line*.

4 The three simplest answers are $\{\mathbf{v}_1, \mathbf{v}_2\}$ or $\{\mathbf{v}_1, \mathbf{v}_3\}$ or $\{\mathbf{v}_2, \mathbf{v}_3\}$. Other answers are possible.

5 the set $\{v_1, v_2\}$, $\{v_2, v_3\}$,$\{v_1, v_3\}$ are linearly independent and thus each form a basis for *H*.

6 *Hint*: Use the Invertible Matrix Theorem.

7 False. Consider the set $\{\mathbf{0}\}$.

8 True. By the Spanning Set Theorem, S contains a basis for V; call that basis S'. Then T will contain more vectors than S'. By Theorem 10, T is linearly dependent.

Part 2

The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank A and dim Nul A.
2. Find bases for Col A and Row A.
3. What is the next step to perform if one wants to find a basis for Nul A?
4. How many pivot columns are in a row echelon form of A^T?

Part 2 solutions

1. A has two pivot columns, so rank $A = 2$. Since A has 5 columns altogether, $\dim \text{Nul } A = 5 - 2 = 3$.
2. The pivot columns of A are the first two columns. So a basis for Col A is

$$\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$$

   The nonzero rows of B form a basis for Row A, namely, $\{(1, -2, -4, 3, -2), (0, 3, 9, -12, 12)\}$. In this particular example, it happens that any two rows of A form a basis for the row space, because the row space is 2-dimensional and none of the rows of A is a multiple of another row. In general, the nonzero rows of an echelon form of A should be used as a basis for Row A, not the rows of A itself.
3. For Nul A, the next step is to perform row operations on B to obtain the reduced echelon form of A.
4. Rank A^T = rank A, by the Rank Theorem, because Col A^T = Row A. So A^T has two pivot positions.

CHAPTER 5 SUPPLEMENTARY EXERCISES

1. Mark each statement as True or False. Justify each answer. In parts (a)–(f), $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a nonzero finite-dimensional vector space V, and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

___ a. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a vector space.

___ b. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans V, then S spans V.

___ c. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ is linearly independent, then so is S.

___ d. If S is linearly independent, then S is a basis for V.

___ e. If Span $S = V$, then some subset of S is a basis for V.

___ f. If $\dim V = p$ and Span $S = V$, then S cannot be linearly dependent.

___ g. A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

___ h. The nonpivot columns of a matrix are always linearly dependent.

___ i. Row operations on a matrix A can change the linear dependence relations among the rows of A.

___ j. Row operations on a matrix can change the null space.

___ k. The rank of a matrix equals the number of nonzero rows.

___ l. If an $m \times n$ matrix A is row equivalent to an echelon matrix U, and if U has k nonzero rows, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $m - k$.

___m. If B is obtained from a matrix A by several elementary row operations, then rank B = rank A.

___ n. If A is $m \times n$ and rank $A = m$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

___ o. If A is $m \times n$ and the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then rank $A = m$.

___ p. A change-of-coordinates matrix is always invertible.

___ q. If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are bases for a vector space V, then the jth column of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the coordinate vector $[\mathbf{c}_j]_{\mathcal{B}}$.

2. Find a basis for the set of all vectors of the form

$$\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix} \quad \text{(Be careful.)}$$

3. Let $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, and W = Span $\{\mathbf{u}_1, \mathbf{u}_2\}$. Find an *implicit* description of W; that is, find a set of one or more homogeneous equations that characterize the points of W. [*Hint:* When is $\mathbf{b}$ in W?]

4. Suppose $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ are specific polynomials that span a two-dimensional subspace H of $\mathbb{P}_5$. Describe how one can find a basis for H by examining the four polynomials and making almost no computations.

5. What would you have to know about the solution set of a homogeneous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss.

6. Let H be an n-dimensional subspace of an n-dimensional vector space V. Explain why $H = V$.

7. Let S be a maximal linearly independent subset of a vector space V. That is, S has the property that if a vector not in S is adjoined to S, then the new set will no longer be linearly independent. Prove that S must be a basis for V. [*Hint:* What if S were linearly independent but not a basis of V?]

8. Let S be a finite minimal spanning set of a vector space V. That is, S has the property that if a vector is removed from S, then the new set will no longer span V. Prove that S must be a basis for V.

Exercises 9–12 develop properties of rank that are sometimes needed in applications. Assume the matrix A is $m \times n$.

9. Show that if P is an invertible $m \times m$ matrix, then rank PA = rank A. [*Hint:* Explain why P is a product of elementary matrices, and show that PA is row equivalent to A.]