Quiz 3

1. Consider $\sum_{j=1}^{n}(-1)^{j-1}\ j^2 = (-1)^{n-1}\ \frac{n(n+1)}{2}$, i.e.

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}\ n^2 \quad = \quad (-1)^{n-1}\ \frac{n(n+1)}{2}. \tag{1}$$

   (a) Show that the statement (1) is true when $n = 3$.

   (b) Use **mathematical induction** to prove the above statement is true whenever n is a positive integer.

Answer:

(a) For $n = 3$,

$$\begin{aligned} 1^2 - 2^2 + 3^2 &= (-1)^{3-1}\ \frac{3(3+1)}{2} \quad (2)\\ 6 &= 6 \quad (3)\end{aligned}$$

and hence, $P(3)$ is true.

(b) Let $P(n)$ be the statement "$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}\ n^2 = (-1)^{n-1}\ \frac{n(n+1)}{2} = 1.$"
We want to prove that $P(n)$ is true for all integer $n \geq 1$.
(I) **Basis step:** Show that $P(1)$ is true. $P(1)$: $1^2 = (-1)^{1-1}\ \frac{1(1+1)}{2}$ which is equal 1 and so $P(1)$ is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 1$.
Assume that $P(k)$ is true:
$1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1}\ k^2 = (-1)^{k-1}\ \frac{k(k+1)}{2}$. ———($\bigstar$) "inductive hypothesis"
We want to show that $P(k+1)$:
"$1^2 - 2^2 + 3^2 - \cdots + (-1)^{k}\ (k+1)^2 = (-1)^{k}\ \frac{(k+1)(k+2)}{2}$."

$$\begin{aligned}\underbrace{1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1}\ k^2}_{(\bigstar)} + (-1)^k\ (k+1)^2 &= (-1)^{k-1}\ \frac{k(k+1)}{2} + (-1)^k\ (k+1)^2 \text{ by } (\bigstar)\\ &= (-1)^{k-1}\ \frac{k(k+1)}{2} + (-1)^k\ (k+1)^2\\ &= (-1)^k(k+1)\left[(-1)^{-1}\frac{k}{2} + (k+1)\right]\\ &= (-1)^k(k+1)\left[-\frac{k}{2} + (k+1)\right]\\ &= (-1)^k(k+1)\frac{(k+2)}{2}\end{aligned}$$

which implies that $P(k+1)$ is true.
Hence, from (I) basis step and (II) inductive step, $P(n)$ is true for any integer $n \geq 1$ by mathematical induction proof. ■