



1. Introduction

1.1 Review of Some Statistical Concepts

The notation \sum (sigma), in mathematical term, denotes the **summation**

$$\sum_{i=1}^n x_i = x_1 + x_2 + \dots + x_n \tag{1.1}$$

The noteworthy properties of summation include:

1. $\sum_{i=1}^n k = nk$ ✓
2. $\sum_{i=1}^n kx_i = k\sum_{i=1}^n x_i$, where k is a constant term.
3. $\sum_{i=1}^n (a + bx_i) = na + b\sum_{i=1}^n x_i$, where a and b are constants.
4. $\sum_{i=1}^n (X_i + Y_i) = \sum_{i=1}^n X_i + \sum_{i=1}^n Y_i$.

Multiple summation is the summation of variable that is in the form of matrix, shown as,

row $\sum_{i=1}^n \sum_{j=1}^m x_{ij}$
column $\sum_{i=1}^n (x_{i1} + x_{i2} + \dots + x_{im}) = (x_{11} + x_{21} + \dots + x_{n1}) + (x_{12} + x_{22} + \dots + x_{n2}) + \dots + (x_{1m} + x_{2m} + \dots + x_{nm})$

$$\tag{1.2}$$

where

$$\mathbf{X} = \begin{matrix}
 \begin{matrix} \color{red}{\text{row}} \\ \color{red}{\text{column}} \end{matrix}
 \begin{matrix}
 \begin{matrix} x_{11} & x_{12} & \dots & x_{1m} \\
 x_{21} & x_{22} & \dots & x_{2m} \\
 \vdots & \vdots & \ddots & \vdots \\
 x_{n1} & x_{n2} & \dots & x_{nm} \end{matrix}
 \end{matrix}
 \end{matrix}
 \begin{matrix}
 \color{green}{\text{column}} \\
 \color{green}{\text{row}}
 \end{matrix}$$

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The significant properties of multiple summations are:

$$① \sum_{i=1}^n \sum_{j=1}^m X_{ij} = \sum_{j=1}^m \sum_{i=1}^n X_{ij}$$

$$2. \sum_{i=1}^n \sum_{j=1}^m X_i Y_j = \sum_{i=1}^n X_i \times \sum_{j=1}^m Y_j$$

$$3. \sum_{i=1}^n \sum_{j=1}^m (X_{ij} + Y_{ij}) = \sum_{i=1}^n \sum_{j=1}^m X_{ij} + \sum_{i=1}^n \sum_{j=1}^m Y_{ij}$$

$$4. (\sum_{i=1}^n X_i)^2 = \sum_{i=1}^n X_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n X_i X_j$$

The product operator \prod is defined as:

$$\prod_{i=1}^n x_i = x_1 * x_2 \dots * x_n \quad (1.3)$$

1.2 Experiment

Sample space is the set of all possible results of an experiment. For example, if you toss the coin twice, all feasible outcomes are composed of head twice, head followed by tail, tail followed by head, and tail twice. Let H and T denotes head and tail, respectively. The sample space can be written as,

$$SS = \{HH, HT, TH, TT\}$$

Sample Point is the member of sample space, eg. the event that head occurs twice from tossing a coin twice. Specifically, sample point is,

$$SP = HH \text{ or } HT \text{ or } TH \text{ or } TT$$

Events are the set of specific consequences of the experiment such as the events that head occurs twice. Events are the subset of sample space.

$$A = \text{the event that head occurs twice} = \{HH\}$$

Events are **mutually exclusive**, if the occurrence of one event makes no other events in sample space possible. As an illustration, for the experiment of tossing two coins once, let C be the event that both turn head and D be the event that both turn tail. Since C and D cannot happen at the same time, these two events are said to be mutually exclusive. Another example is the experiment of drawing one card from the standard 52-card deck, let E be the event that the rank of card is King and F be the event that suit of card is Clubs. As the event E and F can occur simultaneously, namely the King of Clubs, the two events are not mutually exclusive.

Events are **collectively exhaustive** if they cover all possible outcomes in the sample space. With the experiment of tossing the coin twice, let A be the event that head appears twice, B be the event that

tail appears twice, and C be the event that head and tail each appear once. In this case, A, B and C are collectively exhaustive since all events cover all possible results from sample space; that is, HH, HT, TH and TT.

1.3 Probability and Random Variable

OR LIKELIHOOD OR CHANCE

Probability is the possibility that any event will occur, given some specific sample space.

Let A be the event occurring in the given sample space and $P(A)$ be the probability that A will happen. Then, $P(A)$ is defined as;

$$P(A) = \frac{\text{the number of times the event } A \text{ will occur}}{\text{the number of all possible outcomes in sample space}} \quad (1.4)$$

For instance, to draw one card from the standard 52-card deck, let A be the event that the rank of card is 2. Times the event will occur is 4 and the amount of all possible outcomes is 52; hence, the probability of A is $\frac{4}{52}$ or $\frac{1}{13}$.

Some properties of probability are;

- $0 \leq P(A) \leq 1$

- If A , B and C are exhaustive set, then,

$$P(A) + P(B) + P(C) = 1$$

- If A , B and C are mutually exclusive, then,

$$P(A + B + C) = P(A) + P(B) + P(C)$$

Suppose that the results of an experiment are in the form of value, the variable, whose value is determined by one of those results, is known as **Random Variable**. Random variable can be either **discrete** or **continuous value**.

For discrete random variable, the example is the sum of the values on the face of two dice, when rolling two dice once. In other word, the obtained sum will range from 2 to 12, and it is impossible to get 2.5 or 3.5.

For continuous random variable, the example is the height of the high-school student, constricted to the range from 160 to 180 centimetres. It can be seen that the value of the height need not be the integers and can take the value of 160.5 or 160.52 centimetres.

These two distinct characteristics of random variable enable us to classify them into different probability density functions, which would be stated in section 1.4.

1.4 Probability Density Function

PDF

As the value of random variable depends on an experiment, the probability density function would portray the overall image of possible random results. The type of the probability density function relies on the characteristics of the random variable. In this section, many important types are discussed.

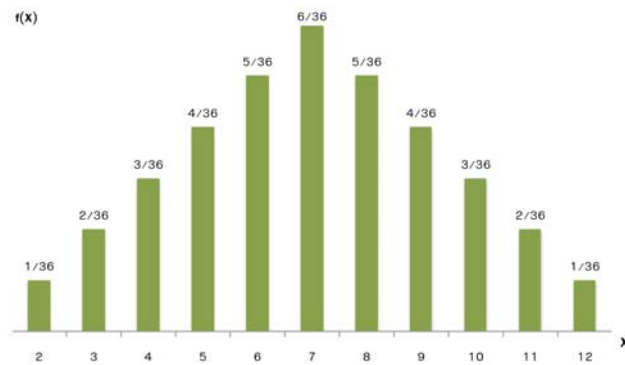
1.4.1 Probability Density Function for Discrete Random Variable

Let X be the discrete random variable with the value x_1, x_2, \dots, x_n and we get,

$$\begin{aligned} f(x) &= P(X = x_i) & \text{for } i &= 1, 2, \dots, n \\ f(x) &= 0 & \text{for } x &\neq x_i \end{aligned}$$

Example: Let X be random variable of the sum of values on the face of two dices. The value might be 2 or 12, that is the value from both rolling round is 1 or 6, respectively. The Figure 1 summarizes all possible results#

Figure 1: Probability Density function of the Sum of Values on the Side of the Dice, Obtained from Rolling the Dice Twice



$$\begin{aligned} E(X) &= \sum x f(x) \\ &= 2 \cdot \left(\frac{1}{36}\right) + 3 \cdot \left(\frac{2}{36}\right) + 4 \cdot \left(\frac{3}{36}\right) \\ &\quad + 5 \cdot \left(\frac{4}{36}\right) + 6 \cdot \left(\frac{5}{36}\right) + 7 \cdot \left(\frac{6}{36}\right) \\ &\quad + 8 \cdot \left(\frac{5}{36}\right) + 9 \cdot \left(\frac{4}{36}\right) + 10 \cdot \left(\frac{3}{36}\right) \\ &\quad + 11 \cdot \left(\frac{2}{36}\right) + 12 \cdot \left(\frac{1}{36}\right) \\ &= 7. \end{aligned}$$

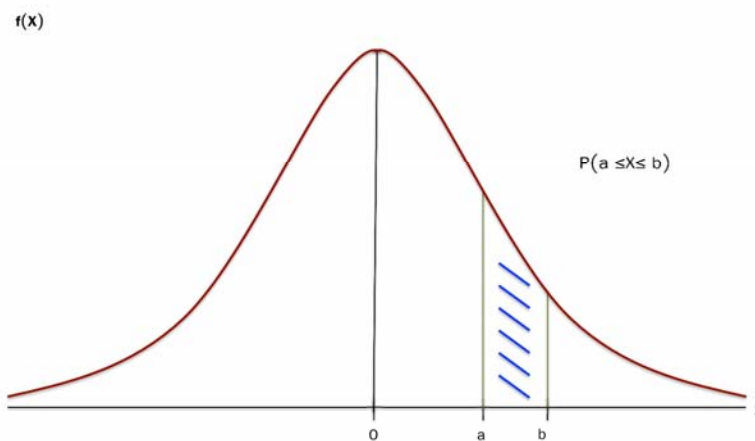
1.4.2 Probability Density Function for Continuous Random Variable

Let X be the continuous random variable. The probability density function of X satisfies the three following conditions.

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $\int_a^b f(x)dx = P(a \leq x \leq b)$

Figure 2 exhibits the probability density function for the continuous random variable, where the area under the curve represents the probability that the variable will lay on that range. Specifically, $P(a \leq X \leq b)$ means the probability that X will take the value between a and b .

Figure 2: Probability Density Function for Continuous Random Variable



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1.4.3 Joint Probability Density Function

In this section, only **joint probability density function** for discrete variable is discussed. Let X and Y be discrete random variables. The joint probability density function, identifying the probability that X and Y happen simultaneously, is written as,

$$f(X, Y) = P(X = x \text{ and } Y = y)$$

Example: The following table explains the joint probability density function.

Table 1: The table illustrating the joint probability density function of X and Y

		X		
		-1	0	1
Y	1	0.11	0.08	0.05
	2	0.09	0.05	0.03
	3	0.35	0.07	0.17

→ $f(Y=1)$
→ $f(Y=2)$
→ $f(Y=3)$

According to the table 1, the probability that random variable X will be 0 and random variable Y will be 3 is 0.07 or 7 percent. In mathematical term, it can be written as $f(X=0, Y=3) = 0.07$.

$$\begin{aligned} f(X=-1) &= 0.11 + 0.09 + 0.35 \\ &= 0.55 \end{aligned}$$

$$\begin{aligned} f(X=0) &= 0.08 + 0.05 + 0.07 = 0.20 \\ f(X=0, Y=1) &= 0.08 \\ f(X=0, Y=2) &= 0.05 \\ f(X=0, Y=3) &= 0.07 \\ f(X=1) &= 0.05 + 0.03 + 0.17 \\ &= 0.25 \end{aligned}$$

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1.4.4 Marginal Probability Density Function

The above joint probability density function $f(X, Y)$ shows the joint distribution of two variables. On the other hand, **marginal probability density function** with respect to joint probability function, displays the probability density function of single variable like $f(X)$, $f(Y)$, which can be derived from;

$$\begin{aligned} f(X) &= \sum_Y f(X, Y) \quad \text{called} \quad \text{marginal PDF of } X \\ f(Y) &= \sum_X f(X, Y) \quad \text{called} \quad \text{marginal PDF of } Y \end{aligned}$$

where \sum_Y or \sum_X means the summation of probability over all values of X and Y respectively.

Example: According to Table 2 above, marginal PDF of X is obtained from

$$\begin{aligned} f(X = -1) &= \\ &= \\ &= \\ &= \\ f(X = 0) &= \sum_Y f(X = 0, Y) \\ &= f(X = 0, Y = 1) + f(X = 0, Y = 2) + f(X = 0, Y = 3) \\ &= 0.08 + 0.05 + 0.07 \\ &= 0.20 \\ f(X = 1) &= \sum_Y f(X = 1, Y) \\ &= f(X = 1, Y = 1) + f(X = 1, Y = 2) + f(X = 1, Y = 3) \\ &= 0.05 + 0.03 + 0.17 \\ &= 0.25 \end{aligned}$$

marginal PDF of Y is obtained from

$$\begin{aligned} f(Y = 1) &= \sum_X f(X, Y = 1) \\ &= f(X = -1, Y = 1) + f(X = 0, Y = 1) + f(X = 1, Y = 1) \\ &= 0.11 + 0.08 + 0.05 \\ &= 0.24 \\ f(Y = 2) &= \sum_X f(X, Y = 2) \\ &= f(X = -1, Y = 2) + f(X = 0, Y = 2) + f(X = 1, Y = 2) \\ &= 0.09 + 0.05 + 0.03 \\ &= 0.17 \\ f(Y = 3) &= \sum_X f(X, Y = 3) \\ &= f(X = -1, Y = 3) + f(X = 0, Y = 3) + f(X = 1, Y = 3) \\ &= 0.35 + 0.07 + 0.17 \\ &= 0.59 \end{aligned}$$

NOTICE THAT

$$\begin{aligned} \sum_Y f(Y) &= f(Y = 1) + f(Y = 2) + f(Y = 3) \\ &= 0.24 + 0.17 + 0.59 \\ &= 1 \quad \# \\ \sum_X f(X) &= f(X = -1) + f(X = 0) + f(X = 1) \\ &= 0.55 + 0.20 + 0.25 \\ &= 1 \quad \#. \end{aligned}$$

According to the calculation above, the result can be summarized into Table 2.

Table 2 shows joint probability of random variable X and Y

		X			
		-1	0	1	
Y	1	0.11	0.08	0.05	$f(Y=1)$ = 0.24
	2	0.09	0.05	0.03	$f(Y=2)$ = 0.17
	3	0.35	0.07	0.17	$f(Y=3)$ = 0.59
		$f(X=-1)$ = 0.55	$f(X=0)$ = 0.20	$f(X=1)$ = 0.25	$f(X)$ = 1

$\rightarrow \sum_Y f(Y) = \underline{\hspace{2cm}}$

1.4.5 Conditional Probability Density Function

Conditional probability density function is the probability of one event given that some events have already occurred. The function is written as,

$f(X|Y) = P(X=x|Y=y)$

This function can be obtained from the joint probability density function through,

$f(X|Y) = \frac{f(X,Y)}{f(Y)}$
 (with handwritten notes: $f(X,Y)$ → JOINT PDF, $f(Y)$ → MARGINAL PDF OF Y)

NOTE

- $f(x,y)$ → JOINT PDF
- $f(x)$ → MARGINAL PDF OF X
- $f(y)$ → MARGINAL PDF OF Y
- $f(x|y)$ → CONDITIONAL PDF

Example: According to Table 2, find $f(X=1|Y=2)$ and $f(Y=2|X=0)$

$f(X=0|Y=1) = \frac{f(X=0, Y=1)}{f(Y=1)} = \frac{0.08}{0.24} = 0.33 \#$

$f(Y=2|X=0) = \frac{f(X=0, Y=2)}{f(X=0)} = \frac{0.05}{0.20} = 0.25 \#$

Example: Let event A be tossing the dice once and the point is odd number and B be the tossing the dice once and the point is at least 5. Find the probability that the point coming up is odd given that the point has to be at least 5.

Answer A and B will occur simultaneously if the point from tossing the dice is 5; so, the joint probability of A and B is $\frac{1}{6}$. The probability that B occurs is $\frac{2}{6}$. Hence, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$$

1.4.6 Statistical Independence

Two random variables are **independent** if the resulting value of one variable does not affect the resulting value of the other; namely,

$$f(X, Y) = f(X)f(Y)$$

$f(x, y)$
 \leftrightarrow JOINT PDF = (MPDF OF X) \times (MPDF OF Y)
 $f(x)$ $f(y)$

Example: Consider Mr. Ake's expenditure for a meal and the Miss Somsri's expenditure for a dessert. Given that they do not know each other, the realization of Mr. Ake's expenditure does not imply the realization of Miss Somsri's expenditure. We can, thus, conclude that the expenditures of these two people are independent#

Example: Consider drawing cards sequentially from the standard 52-card deck without putting it back into the deck. Once the first card is drawn, the probability of drawing the second card will be influenced because the amount of cards in the deck is reduced. In this case, it can be concluded that drawing the first and second card are not independent#

1.5 Expectation, Variance, Covariance and Correlation

1.5.1 Mean or Expected Value

Because the value of random variable hinges on the value of random results of experiment which cannot be determined certainly, statisticians have invented the measures of central tendency of the random variable. One of them is **expected value**, indicating the mean of the random variable.

For discrete random variable, the expected value is calculated by;

$$E(X) = \sum_{i=1}^n x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + \dots + x_n f(x_n)$$

For continuous random variable, the expected value is calculated by,

$$E(X) = \int_a^b x f(x) dx$$

where;

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$E(X)$ is the measure of central tendency of random variable, resulting from repeated trial of experiment.

$\sum_{i=1}^n x_i f(x_i)$ is the average of random variable weighted by the probability corresponding to each value.

a and b are the lowest and highest constant possible respectively.

Example: Find the expected value of rolling two dice once (Figure 1)

$$E(X) = 7.$$

Example:

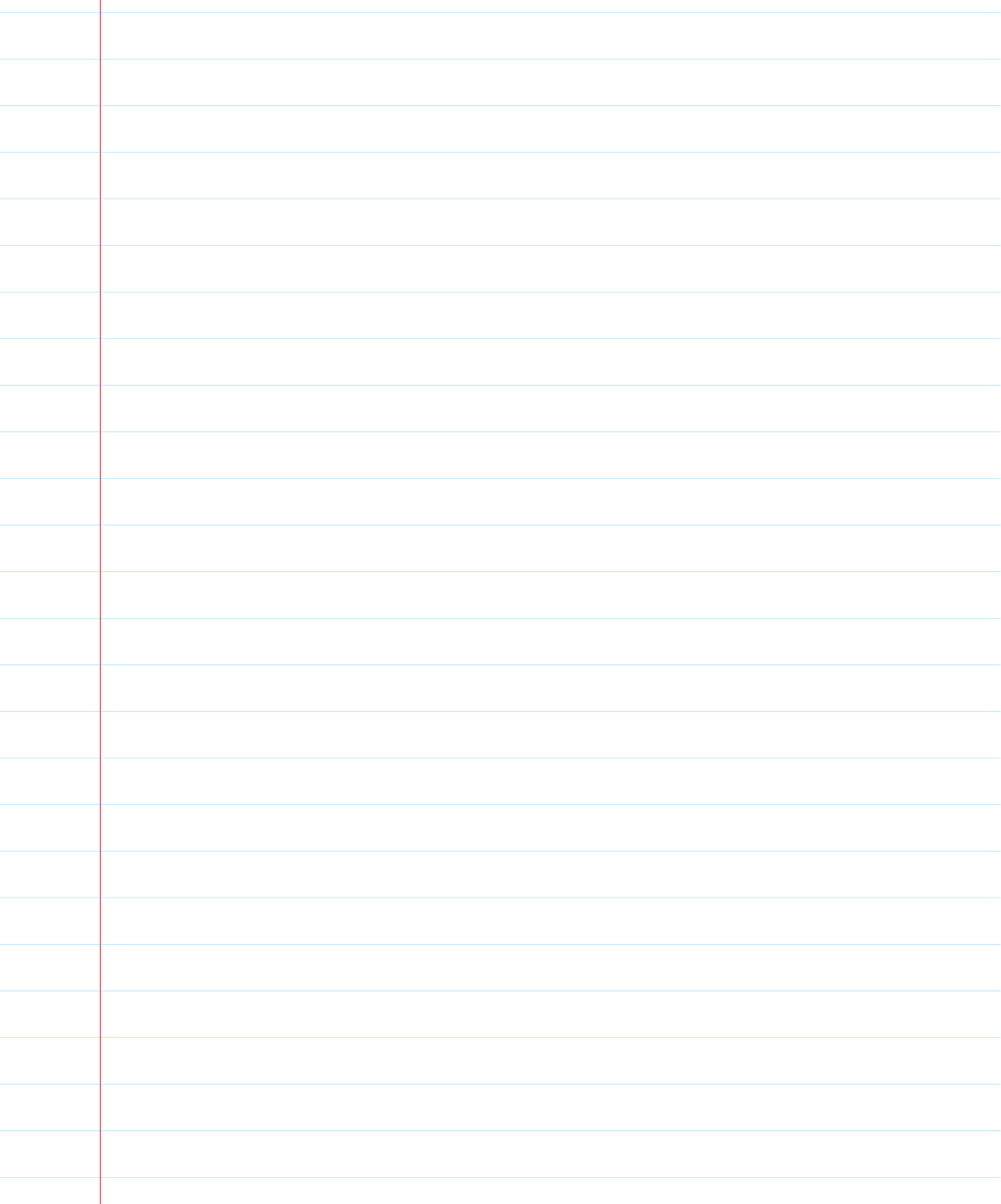
x	-2	1	2
$f(x)$	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

FIND $E(X)$.

$$\begin{aligned} E(X) &= \sum_x x \cdot f(x) = -2 \cdot \left(\frac{5}{8}\right) + 1 \left(\frac{1}{8}\right) + 2 \left(\frac{2}{8}\right) \\ &= -\frac{5}{8} \cdot \# \end{aligned}$$

FIND $E(X^2)$.

$$E(X^2) = \underline{(-2)}^{\textcircled{2}} \left(\frac{5}{8}\right) + \underline{(1)}^{\textcircled{2}} \cdot \left(\frac{1}{8}\right) + \underline{(2)}^{\textcircled{2}} \left(\frac{2}{8}\right)$$



Crucial properties of expected value include:

1. $E(b) = b$
2. $E(aX + b) = aE(X) + b$
3. $E(XY) = E(X)E(Y)$; given that X and Y are independent
4. $E(g(X)) = \sum_x g(X)f(X)$

where a and b are constant.

OR CONDITIONAL MEAN

Conditional expectation value is the expectation value of random variable under some conditions such as expected value of X conditional on Y or $E(X|Y=5)$

Let $f(X, Y)$ be the joint probability function of X and Y . The expectation of X conditional on some X value of Y is defined as,

For discrete random variable $E(X|Y=y) = \sum_X X_i f(X|Y=y)$

		-2	0	2	3
Y	3	.27	.08	.16	0
	6	0	.04	.10	.35

For continuous random variable $E(X|Y=y) = \int_{-\infty}^{\infty} X_i f(X|Y=y)$

Example

$$\begin{aligned}
 E(Y|X=2) &= \sum_y y \cdot f(Y=y|X=2) \\
 &= 3 \cdot f(Y=3|X=2) + 6 \cdot f(Y=6|X=2) \\
 &= 3 \left[\frac{f(X=2, Y=3)}{f(X=2)} \right] + 6 \left[\frac{f(X=2, Y=6)}{f(X=2)} \right] \\
 &= 3 \cdot \left[\frac{0.16}{0.26} \right] + 6 \left[\frac{0.10}{0.26} \right] \\
 &= 4.15
 \end{aligned}$$

WHEN $X=2$, ON AVERAGE, Y IS EQUAL TO 4.15. #

$$\begin{aligned}
 E[Y|X=0] &= \sum_y y \cdot f(Y=y|X=0) \\
 &= 3 \cdot f(Y=3|X=0) + 6 \cdot f(Y=6|X=0) \\
 &= 3 \cdot \frac{f(X=0, Y=3)}{f(X=0)} + 6 \cdot \frac{f(X=0, Y=6)}{f(X=0)} \\
 &= 3 \cdot \left(\frac{0.08}{0.12} \right) + 6 \cdot \left(\frac{0.04}{0.12} \right) \\
 &= \underline{\hspace{2cm}}
 \end{aligned}$$

EXERCISE: FIND $E(X|Y=3)$.

FIND $E(X|Y=6)$.

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1.5.2 Variance

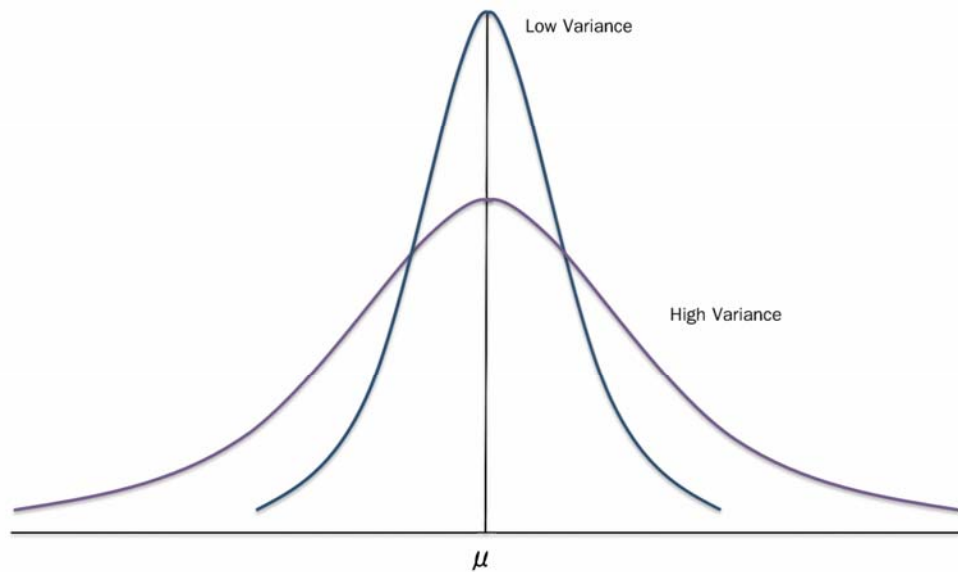
Variance is the measure of dispersion of the value of variable around the expected value. The higher the variance, the more dispersing the random variable (Figure 3). If X is the random variable with expected value μ , we get;

$$\text{Var}(X) = \sigma_X^2 = E[X - E(X)]^2 = E(X)^2 - \mu^2 \quad (1.5)$$

From,

$$\begin{aligned} \text{Var}(X) &= \sigma_X^2 \\ &= E[X - E(X)]^2 \\ &= E[X^2 - 2XE(X) + (E(X))^2] \\ &= E(X^2) - 2(E(X))^2 + (E(X))^2 = E(X^2) - (E(X))^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

Figure 3: Distribution of Random Variables with Different Variance



1.5 Expectation, Variance, Covariance and Correlation

VARIANCE

Important properties of ~~expected value~~ include;

1. $Var(b) = 0$
2. $Var(aX + b) = a^2 Var(X)$
3. $Var(X \pm Y) = Var(X) + Var(Y)$; given that X and Y are independent
4. $Var(aX \pm bY) = a^2 Var(X) + b^2 Var(Y)$

where a and b are constant.

1.5.3 Conditional Variance

The conditional variance of X is given Y = y is defined as following:

$$\begin{aligned}
 var(X|Y = y) &= E\{[X - E(X|Y = y)]^2 | Y = y\} \\
 &= \sum_x [X - E(X|Y = y)]^2 f(x|Y = y) \rightarrow \text{FOR DISCRETE RVs} \\
 &= \int_{-\infty}^{\infty} [X - E(X|Y = y)]^2 f(x|Y = y) dx \rightarrow \text{FOR CONTINUOUS (1.6) RVs}
 \end{aligned}$$

Example COMPUTE $E(Y | X=2)$ AND $VAR(Y | X=2)$.

$E(Y | X=2) = 4.15$ (FROM PREVIOUS EXAMPLE)

	-2	0	2	3
3	0.27	0.08	0.16	0
6	0	0.04	0.10	0.35

$VAR(Y | X=2)$

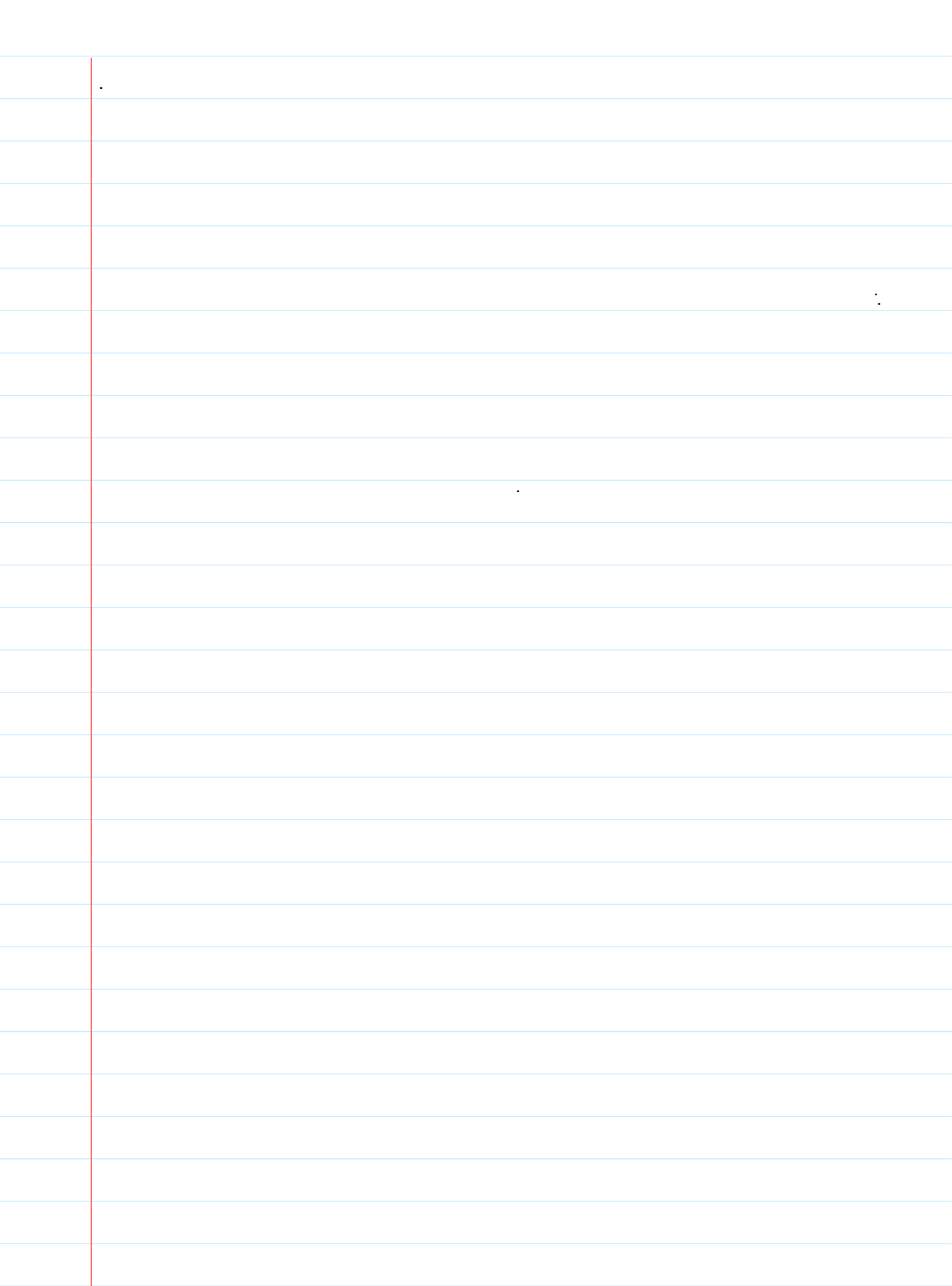
$= \sum_y [Y - E(Y | X=2)]^2 \cdot f(Y | X=2)$

$= (3 - 4.15)^2 \cdot \frac{0.16}{0.26} + (6 - 4.15)^2 \cdot \frac{0.10}{0.26}$

$\frac{f(X=2, Y=3)}{f(X=2)}$

$\frac{f(X=2, Y=6)}{f(X=2)}$

$= \underline{\hspace{2cm}} \cdot \#$



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Properties of conditional expectation and conditional variance

① IF $f(x)$ IS A FUNCTION OF X ,
THEN $E[f(x) | X] = f(x)$

EX: $E[X^3 | X] = E[X^3] = X^3$. #

② IF $f(x)$ AND $g(x)$ ARE FUNCTION OF X

$$E[f(x) \cdot Y + g(x) | X] = f(x) E[Y | X] + g(x)$$

$$E[XY + cX^2 | X]$$

$$= X E[Y | X] + cX^2$$
. #

③ IF X AND Y ARE INDEPENDENT RANDOM VARIABLES,

$$E[Y | X] = E[Y]$$
.

④ $VAR(Y) = E[VAR(Y | X)] + VAR[E(Y | X)]$

EXPECTATION
OF
CONDITIONAL
VARIANCE
OF
Y

VARIANCE
OF
CONDITIONAL
EXPECTATION
OF
Y

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1.5.4 Covariance

Theorem. Let X and Y be two random variables with means μ_x and μ_y , respectively. Then, we can define the covariance between these two variables as following:

$$\text{cov}(X, Y) = E\{(X - \mu_x)(Y - \mu_y)\} = E(XY) - \mu_x \mu_y$$

If X and Y are continuous random variables we can calculate their $\text{cov}(X, Y)$:

$$\begin{aligned} \text{cov}(X, Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X - \mu_x)(Y - \mu_y) f(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} XY f(x, y) dx dy - \mu_x \mu_y \end{aligned}$$

(1.8)

Properties of Covariance

1. If X and Y are independent, the covariance between X and Y is zero.

Proof:
$$\begin{aligned} \text{cov}(X, Y) &= E(XY) - \mu_x \mu_y \\ &= E(X) E(Y) - \mu_x \mu_y \\ &= \mu_x \mu_y - \mu_x \mu_y \\ &= 0. \end{aligned}$$

2. $\text{cov}(a + bX, c + dY) = bd * \text{cov}(X, Y)$, where a, b, c , and d are constants.

Proof:
$$\begin{aligned} \text{cov}(a + bX, c + dY) &= E[(a + bX) - [a + bE(X)]][c + dY - [c + dE(Y)]] \\ &= E[\cancel{a} + bX - \cancel{a} - bE(X)][\cancel{c} + dY - \cancel{c} - dE(Y)] \\ &= E[b(X - E(X)) \cdot d(Y - E(Y))] \\ &= b \cdot d \cdot E[(X - E(X))(Y - E(Y))] \\ &= b \cdot d \cdot \text{cov}(X, Y). \quad \# \end{aligned}$$