4.5) System of simultaneous equations and Matrix approach to the solution

A system of linear equations typically comprises of **M equations and N unknown.** The N-unknown vector of "x" that satisfy all the M equations simultaneously is called the solution of the system.

Our task is to solve for the N-unknown varaibles!

If exist, matrix procedure allows us to solve for the solution under any arbitrary numbers of M and N. However, we would focus only the case when M =N. That is, we concentrate on the *just-identified* system of equation.

Just-(exactly) identified system

System of linear equations: exact identified system

N-Unknown variables

Equation 1: $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = d_1$

Equation 2: $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = d_2$

$\vdots$

Equation N: $a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N = d_N$

Solution to the N-unknown variables can be derived by two methods.

- An inverse matrix approach
- The Cramer's rule

4.5.1) Inverse matrix method

The system can be compactly written in the following matrix representation

$$Ax = d$$

Two approaches
· Inverse Matrix
· Cramer's Rule

A_{NN}

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \\ d_N \end{bmatrix}$$

A = matrix of coefficients that are associated with unknown variables

d = matrix of coefficients that are independent of unknown variables

x = vector of unknown variables

= endogenous variables

Economics Model : All the xs Endogenous Variable.

To obtain the solution, pre-multiply A with an inverse of A.

$$A^{-1}Ax = A^{-1}d$$

By the definition of inverse matrix A, it must be the case that

$$A^{-1}A = I_N$$

→ ~~Inde~~ Identity Matrix

As a result, we yield that

$$x = A^{-1}d$$

Three unknowns
(x, y, z)

$$2x + 3y - 3z = 9$$

$$x + 3z - 3y = 6$$

$$x - z = 9$$

$$\times$$

$$\underbrace{\begin{pmatrix} \underline{2} & \underline{3} & \underline{-3} \\ \underline{1} & \underline{-3} & \underline{3} \\ \underline{1} & \underline{0} & \underline{-1} \end{pmatrix}}_{3\times 3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 9 \end{pmatrix}$$

Finding the inverse of A:

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

$\rightarrow adj(A) = [Cof(A)]^T$

where adjoint of A, i.e. adj(A), is an n x n matrix transpose of the matrix whose entries are each corresponding cofactor.

What is the cofactor matrix?

$Cof(A) \Rightarrow Cof(i,j) = (-1)^{i+j} \cdot M_{ij}$

We need to understand before what *minor* is.

Minor

M_{ij} is the subdeterminant from deleting the i-th row and j-th column.

Example 4.F

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$Cof_{13}(A) = (-1)^{1+3} \cdot M_{13}$

$= (1) \cdot (b_1c_2 - c_1b_2)$

|Matrix delete 1-row/3-colum|

$M_{13} =$

$$\boxed{\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \rightarrow M_{13} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1 \cdot c_2 - c_1b_2}$$

$M_{22} =$

$$\boxed{\begin{pmatrix} a_1 & a_3 \\ c_1 & c_3 \end{pmatrix} \rightarrow M_{22} = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1}$$

$Cof_{22}(A) = (-1)^{2+2} \cdot (a_1c_3 - a_3c_1)$

Cofactor matrix is the matrix whose correspondingly value for the i-th row and j-th column is obtained by,

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Example 4.G: Find the cofactor and the adjoint of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix}$$

cof(A) ; adj(A)

$M_{11}(A) = \begin{vmatrix} 0 & -5 \\ 1 & 1 \end{vmatrix} = +5$

$M_{12}(A) = \begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix} = 13$

$M_{13}(A) = \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3$

$M_{21}(A) = \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -4$

$M_{22}(A) = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$

$M_{23}(A) = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$

$M_{31}(A) = \begin{vmatrix} -1 & 3 \\ 0 & -5 \end{vmatrix} = 5$

$M_{32}(A) = \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} = -19$

$M_{33}(A) = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} = 3$

$C_{11} = (-1)^2 \cdot 5 = 5$

$C_{12} = (-1)^3 \cdot 13 = -13$

$C_{13} = 3$ ~~C_{33}~~

$C_{21} = 4$ $C_{31} = +5$

$C_{22} = -4$ $C_{32} = 19$

$C_{23} = -4$ $C_{33} = 3$

$$\mathrm{cof}(A) = \begin{pmatrix} 5 & -13 & 3 \\ 4 & -4 & -4 \\ 5 & 19 & 3 \end{pmatrix} \Rightarrow \underbrace{\mathrm{adj}(A) = (\mathrm{cof}(A))^T}_{\mathrm{adj}(A)} = \begin{pmatrix} 5 & 4 & 5 \\ -13 & -4 & 19 \\ 3 & -4 & 3 \end{pmatrix}$$

Determinant

For N = 2, determinant of any square matrix A can be calculated by:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

Example 4.H

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

det(A) = |A| = (3)(5) − (2)·(1)
= 15−2 = 13

$$A = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$$

det(A) = |A| = 2·24 − 6·8 = 0

Definition: A is said to a *singular* matrix if det(A) = 0

Theorem:

If A is a singular matrix, the just-identified system contains infinite number of solution.

Unique ⇒ |A| ≠ 0

Lapace expansion

To find the determinant of any square matrix A of order N (N>2), select any row (or column) of A and *multiply each entry in the row (column) by its cofactor*. The *sum of these products* is defined to be the determinant of A and is called a *determinant of order n.*

> **Tip!: Pick any row or column whose entry contains the most of zero.**

Example 4.I: Find the determinant and inverse matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix} \quad A^{-1}\,? = \frac{\operatorname{adj}(A)}{|A|} = \frac{\begin{pmatrix} 5 & 4 & 5 \\ -13 & -4 & 19 \\ 3 & -4 & 3 \end{pmatrix}}{32}$$

|A| = sum of the product between entries in any selected column/row (and) with the associated Cofactor.

2 row

$$|A| = a_{21}\cdot cof_{21}(A) + a_{22}\cdot Cof_{22}(A) + a_{23}\cdot Cof_{23}(A)$$

$$= (3)(-1)^{2+1}\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} + 0\cdot\boxed{} + (-5)(-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 3(-1)(-4) + 0 + (-5)(-1)(4)$$

$$= 12 + 0 + 20 \Rightarrow 32$$

det(A) = 32

4.5.2) Cramer's rule

The linear system $\boldsymbol{Ax} = \boldsymbol{d}$ has a solution given by *Cramer's formulas:*

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1,2,3,\dots,N$$

where (i) $\boldsymbol{x_i}$ are the unknowns of the system or the i-th entries of x, and

(ii) the matrix A_i is obtained from $\boldsymbol{A}$ by replacing the ***i-th*** column by the vector ***d.***

Example 4.J: Find the solution using the Cramer's rule

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$x = \frac{\det(A_x)}{\det(A)} \quad \text{\#1 col}^{n}$$

$$y = \frac{\det(A_y)}{\det A} \quad \text{\#2 col}^{n} \; ; \; z = \frac{\det(A_z)}{\det A} \quad \text{\#3 col}^{n}$$

$$x = \frac{\left|\begin{matrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{matrix}\right| \begin{matrix} 0 & 2 \\ 1 & 1 \\ 0 & 2 \end{matrix}}{\left|\begin{matrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{matrix}\right| \begin{matrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{matrix}} = \frac{\left.\begin{matrix} 0+0+0 \\ -0-0-6 \end{matrix}\right\} = -6}{\begin{matrix} (1)(1)(3)+(2)(1)(1) \\ +(0)(-1)(2) \\ -(0)(1)(1)-(2)(1)(1) \\ -(3)(-1)(2) \\ = 5+4 = 9 \end{matrix}}$$

$$x^{*} = \frac{-6}{9} = -\frac{2}{3} \; \#$$

$$y = \frac{\begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix}}{\det(A) = 9}$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix}}{\det(A) = 9}$$

4.3 Application examples

Note: Mapping the linear economic model in a matrix form

Example 4.K: Using the form Ax = d, write the following market model in the matrix form.

$$Qd = Qs$$

$$Qd = a - bP + cY$$

$$Qs = e + fP + gW$$

Q ; P that clears the market

when a, b, c, e, f, g are parameters. Y = income and W = weather condition.

$$\left.\begin{aligned} Q &= a - bp + cy \\ Q &= e + fp + gw \end{aligned}\right\} Ax = d$$

$$\left.\begin{aligned} Q + bp &= a + cy \\ Q - fp &= e + gW \end{aligned}\right\}$$

$$x = \begin{pmatrix} Q \\ p \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} 1 & b \\ 1 & -f \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} Q \\ p \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} a+cy \\ e+gw \end{pmatrix}}_{d}$$

$$\begin{pmatrix} Q \\ p \end{pmatrix} = \begin{pmatrix} 1 & b \\ 1 & -f \end{pmatrix}^{-1} \begin{pmatrix} a+cy \\ e+gw \end{pmatrix}$$

#

Q = ? ; P = ?