Relations & Functions II

1 Inverse of A Function and Inverse Function

Definition 1.1 (inverse of function)**.** Suppose $f : X \to Y$ is a function. Then we can write

$$y = f(x), x \in X \qquad \text{or} \qquad f = \{(x, y)|x \in X, y = Y\}.$$

The **inverse of function** f, denoted f^{-1}, is defined as

$$\boxed{f^{-1}(y) = x \quad \Leftrightarrow \quad y = f(x)} \qquad \text{or} \qquad \boxed{f^{-1} = \{(y, x)|(x, y) \in f\}}.$$

Note that the inverse of function f may or may not be a function. If f^{-1} is a function f^{-1} is called the **inverse function** of f.

If f is a one-to-one correspondence from a set X to a set Y, then there is a function from Y to X that "undoes" the action of f. That is, it sends each element of Y back to the element of X that it came from. This function is called the inverse function for F.

Theorem 1.1. Suppose $f : X \to Y$ is a one-to-one correspondence. I.e., suppose f is one-to-one and onto. Then there is a function $f^{-1} : Y \to X$ that is defined as follows:
Given any element $y \in Y$,
$f^{-1}(y) =$ the unique element $x \in X$ such that $f(x)$ equals y. I.e.,

$$\boxed{f^{-1}(y) = x \quad \Leftrightarrow \quad y = f(x).}$$

The function f^{-1} is called the **inverse function** for f.

Theorem 1.2. If X and Y are sets and $f : X \to Y$ is one-to-one and onto, then $f^{-1} : Y \to X$ is also one-to-one and onto.

Example 1.1. Let $X = \{1, 2, 3\}$ and $Y = \{-1, -2, -3\}$.
Let $f = \{(1, -3), (2, -1), (3, -3)\}$ be a function from X to Y.
Find the inverse of f, f^{-1} and determine whether it is the inverse function or not.

Example 1.2. Let $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} | y = 2x - 3\}$ be a function. Find the inverse of the function f and determine whether it is the inverse function of f or not.

Example 1.3. The function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(x) = 4x - 1$ for all real numbers x was shown to be one-to-one in the previous examples. Find its inverse function.

Example 1.4. A function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined as, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$F(x, y) = (x + y, x - y)$$

is shown to be a bijective function in the previous example. Determine its inverse function $F^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$.

2 Composite Function

Definition 2.1 (Composite Function)**.** Let $f : X \to \widehat{Y}$ and $g : Y \to Z$ be functions with the property that the range of f is a subset of the domain of g, $\widehat{Y} \subseteq Y$. Define a new function $g \circ f : X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X,$$

where

- $g \circ f$ is read "g circle f " and
- $g(f(x))$ is read "g of f of x."

The function $g \circ f$ is called the **composition** of f and g.

Remarks: Composition of functions is **not a commutative operation**.
For general functions f and g, $f \circ g$ need not necessarily equal $g \circ f$
(although the two may be equal).

Definition 2.2. For $f : X \to \widehat{Y}$ and $g : Y \to Z$ with $\widehat{Y} \not\subseteq Y$, but

$$R_f \cap D_g \neq \emptyset$$

, where R_f is the range of f and D_g is the domain of g ($D_g = Y$). We can still define a new function $g \circ f : \widehat{X} \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in \widehat{X} \subseteq X.$$

Notice that when $\widehat{Y} \subseteq Y$, the domain $\widehat{X}$ of $g \circ f$ will be the same as the domain X of f. However $\widehat{Y} \not\subseteq Y$, but $R_f \cap D_g \neq \emptyset$, $\widehat{X}$ may be smaller than X.
Constructing composite function

Notation Let $f : X \to Y$ be a function.

- D_f denotes the **domain** of the function f.
- R_f denotes the **range** of the function f.

Composite Function
Let f and g be functions such that $\boxed{R_f \cap R_g \neq \emptyset}$. Then we can define a composite function

$$(g \circ f)(x) = g(f(x)), \quad x \in \widehat{X} \subseteq D_f.$$

- Domain of $g \circ f$: $D_{g \circ f} \subseteq D_f$.
- Range of $g \circ f$: $R_{g \circ f} \subseteq R_g$.
- If $R_f \subseteq D_f$, then $D_{g \circ f} = D_f$.

Example 2.1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the successor function and let $g : \mathbb{Z} \to \mathbb{Z}$ be the squaring function. Then $f(n) = n + 1$ for all $n \in \mathbb{Z}$ and $g(n) = n^2$ for all $n \in \mathbb{Z}$.

- Find the compositions $g \circ f$ and $f \circ g$.
- Is $g \circ f = f \circ g$? Explain.

Example 2.2. Let $X = \{1, 2, 3\}$, $\widehat{Y} = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Define functions $f : X \to \widehat{Y}$ and $g : Y \to Z$ by

- $f(1) = c$, $f(2) = b$, $f(3) = a$,
- $g(a) = y, g(b) = y, g(c) = z, g(d) = z, g(e) = z$.

Draw the arrow diagrams for f, g, and $g \circ f$.
Determine the range for each of f, g, and $g \circ f$.

Example 2.3. Let $f(x) = \frac{2}{\sqrt{x-1}}$ and $g(x) = x^2 + 3$. Find $g \circ f$ and $f \circ g$. Find $(g \circ f)(2)$.

Example 2.4. (Composition with the Identity Function):
Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f : X \to Y$ is given by

$$f(a) = u, \quad f(b) = v, \quad f(c) = v, \quad f(d) = u.$$

Let $I_X : X \to X$ and $I_Y : Y \to Y$ with $I_X(x) = x$ and $I_Y(y) = y$ be identity functions. Find $f \circ I_X$ and $I_Y \circ f$.

Theorem 2.1. Composition with an Identity Function
If f is a function from a set X to a set Y , and I_X is the identity function on X, and I_Y is the identity function on Y, i.e. ,

$$I_X(x) = x \quad \forall x \in X, \qquad I_Y(y) = y \quad \forall y \in Y$$

then

$$\boxed{f \circ I_X = f \qquad \text{and} \qquad I_Y \circ f = f.}$$

Let f be a function from a set X to a set Y, and suppose f has an inverse function f^{-1}. Recall that f^{-1} is the function from Y to X with the property that

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

What happens when f is composed with f^{-1}? Or when f^{-1} is composed with f ?

Example 2.5. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Define $f : X \to Y$ by

$$f(a) = z, \quad f(b) = x, \quad f(c) = y.$$

1. Draw the arrow diagram for f.
2. Show that the inverse function f^{-1} exists for the function f.
3. Find f^{-1} and draw the arrow diagram for f^{-1}.
4. Find and draw the arrow diagrams for $f^{-1} \circ f$ and $f \circ f^{-1}$. Compare with I_X and I_Y.

Theorem 2.2. Composition of a Function with Its Inverse
If $f : X \to Y$ is a one-to-one and onto function with inverse function $f^{-1} : Y \to X$, then

$$f^{-1} \circ f = I_X \qquad \text{and} \qquad f \circ f^{-1} = I_Y.$$

Example 2.6. Show that the following statement is true.

Theorem 2.3. The composition of two injective(one-to-one) functions is also injective If $f : X \to Y$ and $g : Y \to Z$ are both **one-to-one** functions, then $g \circ f$ is **one-to-one**.

Example 2.7. Show that the following statement is true.

Theorem 2.4. The composition of two surjective(onto) functions is also surjective(onto) If $f : X \to Y$ and $g : Y \to Z$ are both **onto** functions, then $g \circ f$ is **onto**.

3 Exercise

1. Let $f(x) = x^2 + 5$ and $g(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{1}{x}, & 2 \leq x < 3. \end{cases}$ Find $f \circ g$ and $g \circ f$.

2. Define a function $g : [0, 3) \to \mathbb{R}$ and $h : [0, 3) \to \mathbb{R}$ as

$$g(x) = \frac{1}{x+1}, \quad \text{and} \quad h = \frac{12}{x+1} - 5,$$

and define a function $f : [-1, \infty) \to [0, \infty)$ be a function

$$f(x) = \begin{cases} x + 1, & x \in [-1, 4) \\ \sqrt{x} + 3, & x \in [4, \infty). \end{cases}$$

   (a) Find domains, co-domains, and ranges of f, g and h.

   (b) Find the composite functions $f \circ g$, $g \circ f$ and $f \circ h$ together with their domains and ranges.

3. Define a function $f : [-1, \infty) \to [0, \infty)$ be a function $f(x) = \begin{cases} x + 1, & x \in [-1, 4) \\ \sqrt{x} + 3, & x \in [4, \infty). \end{cases}$

   Show that f is injective and find its inverse function f^{-1}.