The Logic of Quantified Statements: I
TU152: Fundamental Mathematics

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Definition

- A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
- The **domain of a predicate variable** is the set of all values that may be substituted in place of the variable in the *predicate*.

Example: Let $Q(x, y) : x = y + 3$ with domain the collection of numbers $0, 1, 2, 3 \ldots$.
What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?
Answer:

Truth Set of a Predicate

Example: Let $P(x)$ be the predicate "$x^3 > x$" with domain the set **R** of all real numbers. Write

- $P(1)$ and $P(-1)$
- $P(\frac{1}{3})$
- $P(3)$

and indicate which of these statements are true and which are false.

Truth Set of a Predicate

Definition

- If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

  The **truth set** of $P(x)$ is denoted

  $$\{x \in D | P(x)\}$$

  which is read "the set of all x in D such that $P(x)$."

Example Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

(i) the domain of n is the set $\mathbb{Z}^+$ of all positive integers

(ii) the domain of n is the set $\mathbb{Z}$ of all integers.

Answer:

Notation: $\mathbb{R}, \mathbb{Z}, \mathbb{N}$

Universal Quantifier: $\forall$

Another way to generate propositions is by means of quantifiers. Quantifiers are words that refer to quantities such as “some” or “all” and tell for how many elements a given predicate is true

Definition

Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form

$$\forall x \in D, Q(x),$$

where the symbol $\forall$ denotes “for all” or “for any.”

- It is defined to be **true** if, and only if, $Q(x)$ is true for **every** x in D.
- It is defined to be **false** if, and only if, $Q(x)$ is false for **at least one** x in D.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example:

(i) Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, \quad x^2 \geq x.$$

Show that this statement is true.

(ii) Consider the statement

$$\forall x \in \mathbb{R}, \qquad x^2 \geq x.$$

Find a counterexample to show that this statement is false.

Answer:

The Existential Quantifier: $\exists$

Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form

$$\exists x \in D \quad \text{such that} \quad Q(x).$$

where the symbol $\exists$ denotes "there exists."

- It is defined to be **true** if, and only if, $Q(x)$ is **true** *for at least one* x in D.
- It is **false** if, and only if, $Q(x)$ is **false** for all x in D.

Example:
Consider the statement

$$\exists x \in \mathbb{Z}^+ \quad \text{such that} \quad x^2 = x.$$

Show that this statement is true.
Answer:

Example: Let $D = \{5, 6, 7, 8\}$. Determine the truth value of the following statements:

$$\exists x \in D \quad \text{such that} \quad x^2 = x.$$

Answer:

Example: Let $P(x)$ denote the statement $x > 3$: Determine the truth value of the statement:

$$\exists x \in \mathbb{R}, P(x).$$

Answer:

Example:

Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the universal conditional statement:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boxed{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently,

$$\forall x, P(x) \rightarrow Q(x).$$

- The notation $\boxed{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently,

$$\forall x, P(x) \leftrightarrow Q(x).$$

Example:

Consider the two predicates $P(x)$: x is a factor of 4 and Q(x) : x is a factor of 8. Show that $P(x) \Rightarrow Q(x)$.

Negation of a Universal Statement

Theorem (Negation of a Universal Statement)

The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D, \sim Q(x).$$

Symbolically,

$$\sim (\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x).$$

The negation of a universal statement (“all are”) is logically equivalent to an existential statement (“some are not” or “there is at least one that is not”).

Example:

Suppose the statement

All mathematicians wear glasses

is false. So a correct negation is its negation :

There is at least one mathematician who does not wear glasses.

Negation of an Existential Statement

Theorem (Negation of an Existential Statement)

The negation of a statement of the form

$$\exists x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim Q(x).$$

Symbolically,

$$\sim (\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x).$$

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Example: Rewrite the following statement formally. Then write formal and informal negations.

No politicians are honest.

Answer:

- Formal version:
- Formal negation:
- Informal negation:

Negation of Universal & Existential Statements

Example: Write formal negations for the following statements:

- $\forall$ primes p, p is odd.
- $\exists$ a triangle T such that the sum of the angles of T equals 200°.

Answer

Negation of a Universal Conditional Statement

Negation of a Universal Conditional Statement

$$\sim (\forall x, P(x) \to Q(x)) \equiv \exists x, P(x) \wedge \sim Q(x)$$

Example Write the negation of each of the following statements:

i $\forall x \in \mathbb{R}; x > 3 \to x^2 > 9$

ii Every polynomial function is continuous.

iii There exists a triangle with the property that the sum of angles is greater than 180°:

Answer:

Contrapositive, Converse, and Inverse of a Universal Conditional Statement

Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

- Its contrapositive is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
- Its converse is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
- Its inverse is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example:

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Necessary and Sufficient Conditions, Only If

The definitions of necessary, sufficient, and only if can also be extended to apply to universal conditional statements.

Definition

- $\forall x, r(x)$ is a sufficient condition for $s(x)$, means $\forall x, r(x) \rightarrow s(x)$
- $\forall x, r(x)$ is a necessary condition for s(x) means $\forall x, \sim r(x) \rightarrow \sim s(x)$ or $\forall x, s(x) \rightarrow r(x)$.
- $\forall x, r(x)$ only if $s(x)$ means $\forall x, \sim s(x) \rightarrow \sim r(x)$ or, equivalently, $\forall x, r(x) \rightarrow s(x)$.