Assignment 3 (Pannatat Pongthiengtham 6204640731)

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E[mX_a] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

$$P_s = \pi_s m_s (1) \quad ; \quad X_s = 1$$

$$m_s = \frac{\delta U'(C_s^*)}{U'(C_0^*)} = \frac{\delta\left(\frac{1}{C_s^*}\right)}{\frac{1}{C_0^*}} = \frac{\delta C_0^*}{C_s^*}$$

$$\therefore P_s = \pi_s \left(\frac{\delta C_0^*}{C_s^*}\right)$$

3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E[m_{01} R_s]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d(C_s/C_0)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U(C_0) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

$$m_{01} = \frac{\delta U'(C_1)}{U'(C_0)} = \delta\left(\frac{C_1}{C_0}\right)^{\gamma-1}$$

FOC:

$$1 = \delta\left(\frac{C_s}{C_0}\right)^{\gamma-1} \pi_s R_s$$

$$0 = \delta(\gamma-1)\pi_s R_s \left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right) + \delta\left(\frac{C_s}{C_0}\right)^{\gamma-1} \pi_s \, dR_s$$

$$0 = \delta\pi_s \left((\gamma-1)R_s\left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right) + \left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s\right)$$

$$-\left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s = (\gamma-1)R_s\left(\frac{C_s}{C_0}\right)^{\gamma-2} d\left(\frac{C_s}{C_0}\right)$$

$$-\left(\frac{C_s}{C_0}\right)^{\gamma-1-\gamma+2} = (\gamma-1)R_s \frac{d\left(\frac{C_s}{C_0}\right)}{dR_s}$$

$$\frac{C_s}{C_0} = (1-\gamma)R_s \frac{d\left(\frac{C_s}{C_0}\right)}{dR_s}$$

$$\frac{1}{1-\gamma} = \frac{R_s}{\frac{C_s}{C_0}} \frac{d\left(\frac{C_s}{C_0}\right)}{dR_s}$$

$$\therefore \varepsilon^I = \frac{1}{1-\gamma}$$

4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

$$P = \begin{bmatrix} \frac{1}{R_f} \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{1.05} \\ 6 \end{bmatrix} , \quad X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix} , \quad X^{-1} = \frac{1}{-5}\begin{bmatrix} 5 & -10 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

$$[P_1 \quad P_2] = P^T X^{-1} = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}$$

$$= [0.2476 \quad 0.7048]$$

$\therefore$ Risk-neutral probabilities ($\hat{\pi}_s = P_s R_f$)

$\hat{\pi}_1 = P_1 R_f = (0.2476)(1.05) = 0.26$

$\hat{\pi}_2 = P_2 R_f = (0.7048)(1.05) = 0.74$

b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

$$m_s = \frac{P_s}{\pi_s} \rightarrow m_1 = \frac{P_1}{\pi_1} = \frac{0.2476}{0.5} = 0.4952$$

$$m_2 = \frac{P_2}{\pi_2} = \frac{0.7048}{0.5} = 1.4096$$

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

6.a)

$$E[m(R_i - R_f)] = 0$$

$$E[(a + bR_m)(R_i - R_f)] = 0$$

$$E[aR_i - aR_f + bR_iR_m - bR_mR_f] = 0$$

$$aE[R_i] - aR_f + bE[R_mR_i] - bR_f E[R_m] = 0$$

$$a(E[R_i] - R_f) + b(E[R_m] + E[R_i] + cov[R_m, R_i] - R_f E[R_m]) = 0$$

$$(E[R_i] - R_f)(a + bE[R_m]) + b\,cov[R_m, R_i] = 0$$

$$E[R_i] - R_f = \frac{-b\,cov[R_m, R_i]}{a + bE[R_m]}$$

$$E[R_i] - R_f = \frac{-cov[R_m, R_i]}{\sigma_m^2} \cdot \frac{b\sigma_m^2}{a + bE[R_m]}$$

$$E[R_i] - R_f = -\beta_i \cdot \frac{b\sigma_m^2}{a + bE[R_m]}$$

from CAPM: $E[R_i] - R_f = \beta_i \gamma$

$$\therefore \quad \gamma = \frac{-b\sigma_m^2}{a + bE[R_m]}$$

6.b) Risk free :
$$\frac{1}{R_f} = E[m] = E[a+bR_m] = a + bE[R_m] \quad — ①$$

Market portfolio : $1 = E[mR_m] = E[(a+bR_m)(R_m)] = aE[R_m] + bE[R_m^2]$

from $\sigma_m^2 = E[R_m^2] - (E[R_m])^2$

$$1 = aE[R_m] + b\left(\sigma_m^2 + (E[R_m])^2\right) \quad — ②$$

Rewritten ①

$$a = \frac{1}{R_f} - bE[R_m] \quad — ③$$

③ → ②

$$1 = \left(\frac{1}{R_f} - bE[R_m]\right)E[R_m] + b\left(\sigma_m^2 + (E[R_m])^2\right)$$

$$1 = \frac{E[R_m]}{R_f} + b\sigma_m^2$$

$$b = \left(1 - \frac{E[R_m]}{R_f}\right)\frac{1}{\sigma_m^2} = \frac{-(E[R_m] - R_f)}{R_f\sigma_m^2}$$

Put b → ③

$$a = \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f\sigma_m^2}$$

6.c) From 6.a) → $\gamma = \frac{-b\sigma_m^2}{a + bE[R_m]}$

$$\gamma = \frac{\frac{(E[R_m] - R_f)}{R_f\cancel{\sigma_m^2}} \cdot \cancel{\sigma_m^2}}{\frac{\sigma_m^2 + \cancel{E[R_m](E[R_m] - R_f)}}{R_f\sigma_m^2} - \frac{\cancel{E[R_m](E[R_m] - R_f)}}{R_f\sigma_m^2}}$$

$$\gamma = \frac{\frac{E[R_m] - R_f}{R_f}}{\frac{\cancel{\sigma_m^2}}{\cancel{R_f}\cancel{\sigma_m^2}}}$$

$$\gamma = E[R_m] - R_f$$