Chapter 5 Inverses

Definition 5.1 Let **A** be a square matrix in $\boldsymbol{R}^{n\times n}$. The ***inverse of matrix*** **A**, denoted by $\mathbf{A}^{-1} \in \boldsymbol{R}^{n\times n}$, is matrix with the property that,

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}.$$

If $\mathbf{A}^{-1}$ exists, then **A** is called ***invertible***.

Example Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$. Then **B** is an inverse of **A** because,

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}.$$

Thus, we can write $\mathbf{B} = \mathbf{A}^{-1}$, and equivalently $\mathbf{A} = \mathbf{B}^{-1}$.

That is, from the definition of inverse, if matrix **B** is an inverse of matrix **A**, then matrix **A** is also an inverse of matrix **B**.

In showing that matrix **B** is an inverse of matrix **A**, it will be shown in the next theorem that it is sufficient to show just either $\mathbf{BA} = \mathbf{I}$, or $\mathbf{AB} = \mathbf{I}$. To show this, we need the following definitions of left and right inverses.

Definition 5.2 Let **A**, **B** and **C** be square matrices in $\boldsymbol{R}^{n\times n}$. Then the matrix **B** is a ***left inverse*** of **A** if $\mathbf{BA} = \mathbf{I}$, and the matrix **C** is a ***right inverse*** of **A** if $\mathbf{AC} = \mathbf{I}$.

Theorem 5.1 Let **A**, **B** and **C** be matrices in $\boldsymbol{R}^{n\times n}$. If the matrix **B** is a ***left inverse*** of **A** while the matrix **C** is a ***right inverse*** of **A**, then $\mathbf{B} = \mathbf{C} = \mathbf{A}^{-1}$.

Proof By definition of inverse, it is sufficient to show just that $\mathbf{B} = \mathbf{C}$. By the assumption,

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}. \square$$

By this theorem, we also conclude that if **A** has an inverse, this inverse is the only inverse of **A**.

Corollary 5.1 The inverse of matrix **A**, if exists, is unique.

Proof Directly by Theorem 5.1.❑

Example As we have seen in the proof of Theorem 3.5, for any elementary matrix $\mathbf{E}$, there exists another elementary matrix $\mathbf{F}$ such that $\mathbf{FE} = \mathbf{I}$. Then by Theorem 5.1, we also have $\mathbf{EF} = \mathbf{I}$, and so $\mathbf{E}^{-1} = \mathbf{F}$ and $\mathbf{F}^{-1} = \mathbf{E}$.

Problem Leon [1994], # 18 a, 19, page 62.

19. Let $\mathbf{A}$ be a nonsingular matrix and let $\mathbf{B}$ be an $n \times r$ matrix. Show that the reduced row echelon form of $\begin{bmatrix}\mathbf{A} & \mathbf{B}\end{bmatrix}$ is $\begin{bmatrix}\mathbf{I} & \mathbf{C}\end{bmatrix}$, where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$.

5.1 Properties of Inverse

The following two theorems are additional properties of inverses.

Theorem 5.2 Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices in $\boldsymbol{R}^{n\times n}$. If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Proof By Theorem 5.1, it is sufficient to check if $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$. This is left as an exercise.❑

Corollary 5.2

a) If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k \in \boldsymbol{R}^{n\times n}$ are invertible, then

$$(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1}\cdots\mathbf{A}_1^{-1}.$$

b) If $\mathbf{A}$ is invertible, then $(\mathbf{A}^m)^{-1} = (\mathbf{A}^{-1})^m$.

Proof By Theorem 5.2 and simple induction.❑

Theorem 5.3 Let $\mathbf{A}$ be invertible. Then,

a) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

b) $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.

c) $|\mathbf{A}^{-1}| = \dfrac{1}{|\mathbf{A}|}$.

Proof **a)** Since $\mathbf{A}^{-1}(\mathbf{A}^{-1})^{-1} = \mathbf{I}$, premultiplying both sides by $\mathbf{A}$, we have,

$$\left(\mathbf{AA}^{-1}\right)\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$
$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}.$$

b) Since $\mathbf{I}^{\mathrm{T}} = \left(\mathbf{A}^{-1}\mathbf{A}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\left(\mathbf{A}^{-1}\right)^{\mathrm{T}} = \mathbf{I}$, premultiplying both sides of the last equality by $\left(\mathbf{A}^{\mathrm{T}}\right)^{-1}$ and we have the required result.

c) $1 = |\,\mathbf{I}\,| = \left|\mathbf{AA}^{-1}\right| = |\mathbf{A}|\left|\mathbf{A}^{-1}\right|$.❑

Problem Show that $\left(c\mathbf{A}\right)^{-1} = \frac{1}{c}\mathbf{A}^{-1}$.

Problem **Leon** [1994], #21, page 62. If a symmetric matrix **A** is invertible, then its inverse is also symmetric.

Problem **Fraleigh & Beauregard** [1995], page 86, #38.
38. Let **A** be an invertible square matrix. Recalling Lemma 4.1 and that $\left(\mathbf{BA}\right)^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$, answer the following questions:

- **a)** If two rows of **A** are interchanged, how does the inverse of the resulting matrix compare with ?
- **b)** Answer the question in part **(a)** if, instead, a row of **A** is multiplied by a nonzero scalar *r*.
- **c)** Answer the question in part **(a)** if, instead, *r* times the i^{th} row of **A** is added to the j^{th} row.

5.2 Computation of Inverses

What we have discussed so far are the properties of the inverse of a matrix. We will now find such an inverse matrix, and more importantly, what are the conditions that guarantee the existence of the inverse of a matrix.

If we have a computational algorithm to compute the inverse of a matrix, one of its obvious use is to solve systems of linear equations. If $\mathbf{Ax} = \mathbf{b}$ is a system of *n* linear equations of *n* variables, then **A** is square. Suppose exists, then the solution **x** is given by,

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$$
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Therefore, the calculation of $\mathbf{A}^{-1}$ is equivalent to solving $\mathbf{Ax} = \mathbf{b}$.

The two usual ways of computing $\mathbf{A}^{-1}$ are by using elementary row operations and by the adjoint matrix.

Theorem 5.4 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$. If there exists a series of *k* elementary row operations, each represented by premultiplying **A** by an elementary matrix $\mathbf{E}_i$, $i = 1,2,\ldots,k$, such that when performed on **A** we have $\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1\mathbf{A} = \mathbf{I}$, then $\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1$.

Proof Let $\mathbf{B} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1$. Then **B** is the inverse of **A** by Theorem 5.1.❑

In practice, we write the augmented matrix $= \begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix}$, where the first *n* columns are just columns of **A** and the last *n* columns constitute an identity matrix. Then perform the elementary row operations on until the first *n* columns becomes an identity matrix.

Example Find the inverse of the matrix $\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$. We have,

$$\hat{\mathbf{A}} = \left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 2 & 4 & 0 & 1\end{array}\right] \overset{R_2-2R_1}{\rightarrow} \left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 0 & -2 & -2 & 1\end{array}\right]$$

$$\overset{-\frac{1}{2}R_2}{\rightarrow} \left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 0 & 1 & 1 & -0.5\end{array}\right] \overset{R_1-3R_2}{\rightarrow} \left[\begin{array}{cc|cc}1 & 0 & -2 & 1.5\\ 0 & 1 & 1 & -0.5\end{array}\right].$$

The inverse is $\begin{bmatrix}-2 & 1.5\\ 1 & -0.5\end{bmatrix}$.

Problem Can we find $\mathbf{A}^{-1}$ by performing elementary column operations on $\tilde{\mathbf{A}} = \begin{bmatrix}\mathbf{A}\\ \mathbf{I}\end{bmatrix}$ until we obtain $\begin{bmatrix}\mathbf{I}\\ \mathbf{B}\end{bmatrix}$ and say $\mathbf{B} = \mathbf{A}^{-1}$?

5.2.2 Computing Inverses by Adjoint Matrix

Definition 5.3 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$. The ***adjoint of*** **A**, denoted by ***adj*** **A**, is a matrix in and given by

$$\begin{aligned}\text{adj }\mathbf{A} &= \left[\Delta_{ij}\right]^{\mathrm{T}} \\ &= \begin{bmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{n1} & \Delta_{n2} & \cdots & \Delta_{nn} \end{bmatrix}^{\mathrm{T}},\end{aligned}$$

where Δ_{ij} is the $(i,j)^{\text{th}}$ cofactor of **A**.

Theorem 5.5 Let **A** be a nonsingular matrix. The inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\text{adj }\mathbf{A}.$$

Proof Since the cofactor $\Delta_{ij} = (-1)^{i+j} M_{ij}$, the determinant of **A** can be written as,

$$|\mathbf{A}| = \begin{cases} \sum_{j=1}^{n} a_{ij}\Delta_{ij}, & i = 1,2,\ldots,n, \\ \sum_{i=1}^{n} a_{ij}\Delta_{ij}, & j = 1,2,\ldots,n. \end{cases}$$

Now by the form of the inverse $\mathbf{A}^{-1}$ given in the theorem,

$$\begin{aligned}\mathbf{AA}^{-1} &= \frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1} & \Delta_{n2} & \cdots & \Delta_{nn} \end{bmatrix}^{\mathrm{T}} \\ &= \frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} \end{bmatrix} \\ &= \frac{1}{|\mathbf{A}|}\begin{bmatrix} \sum_{j=1}^{n} a_{1j}\Delta_{1j} & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{n} a_{2j}\Delta_{2j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{n} a_{nj}\Delta_{nj} \end{bmatrix} \\ &= \mathbf{I}.\end{aligned}$$

The last two equalities are due to the properties of the determinant stated in Theorem 4.1 and 4.2.❑

Problem Verify that

$$\sum_{k=1}^{n} a_{ik}\Delta_{jk} = \begin{cases} |\mathbf{A}|, \text{ if } i = j, \\ 0, \text{ if } i \neq j. \end{cases}$$

Problem Fraleigh & Beauregard [1995], page 272, #35 (*g*)-(*i*), 36-39.

35. Let **A** be a square matrix. Mark each of the following True or False.

_____**g.** The product of a square matrix and its adjoint is the identity matrix.

_____**h.** The product of a square matrix and its adjoint is equal to some scalar times the identity matrix.

_____**i.** The transpose of the adjoint of **A** is the matrix of cofactors of **A**.

36. Prove that the inverse of a nonsingular upper-triangular matrix is upper triangular.
37. Prove that a square matrix is invertible if and only if its adjoint is an invertible matrix.
38. Let **A** be an matrix. Prove that $|\text{adj } \mathbf{A}| = (|\mathbf{A}|)^{n-1}$.
39. Let **A** be an invertible matrix with . Using Exercises 37 and 38, prove that $\text{adj}(\text{adj } \mathbf{A}) = (|\mathbf{A}|)^{n-2}\mathbf{A}$.

Problem Fraleigh & Beauregard [1995], page 85, #26 and 30.

26. Let **A** be a matrix such that $\mathbf{A}^2$ is invertible. Prove that **A** is invertible.
30. A square matrix **A** is said to be idempotent if $\mathbf{A}^2 = \mathbf{A}$.
    - **a)** Give an example of an idempotent matrix other than **0** and **I**.
    - **b)** Show that, if a matrix **A** is both idempotent and invertible, then $\mathbf{A} = \mathbf{I}$.

5.3 Inverse and Determinants

Theorem 5.6 A matrix **A** in $\boldsymbol{R}^{n\times n}$ is ***invertible*** if, and only if, **A** is nonsingular.

Proof We have to show that $\mathbf{A}^{-1}$ exists if, and only if, $|\mathbf{A}| \neq 0$. Suppose $|\mathbf{A}| \neq 0$, then $\mathbf{A}^{-1}$ can be computed using the adjoint of **A**. Therefore $\mathbf{A}^{-1}$ exists.

Now suppose $\mathbf{A}^{-1}$ exists but $|\mathbf{A}| = 0$. Since $\mathbf{A}^{-1}$ exists we can compute its determinant $|\mathbf{A}^{-1}|$ to be some real number. However, by Theorem 5.3 part **(c)**, $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$, which is undefined as $|\mathbf{A}| = 0$. This is a contradiction. So $|\mathbf{A}| \neq 0$.❑

Similar to Corollary 4.3 in the previous chapter, we can identify a system of *n* linear equations with *n* variables as having unique solution by the coefficient matrix **A** being full rank, nonsingular or invertible.

Corollary 5.3 Let **A** be matrix in . The following statements are equivalent:

1. $\mathbf{A}^{-1}$ exists.
2. $|\mathbf{A}| \neq 0$, i.e., nonsingular.
3. rank $\mathbf{A}$ = rank $\mathbf{A}^{\mathrm{T}} = n$.
4. $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any given vector **b**.

Proof By Theorem 5.6 and Corollary 4.3.❑

Problem Write a corollary summarizing results when $\mathbf{A}^{-1}$ does not exist.

Problem Fraleigh & Beauregard [1995], page 85, #25.

25. **a)** If **A** is invertible, is $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$ always invertible?
    b) If **A** is invertible, is $\mathbf{A} + \mathbf{A}$ always invertible?

Problem Johnson, Riess & Arnold [1998], page 103, #67, 68, 70, and 71.

67. Suppse that $\mathbf{AB} = \mathbf{0}$, where **A** is nonsingular. Prove that $\mathbf{B} = \mathbf{0}$.
68. Let **A**, **B**, and **C** be matrices such that **A** is nonsingular and $\mathbf{AB} = \mathbf{AC}$. Prove that $\mathbf{B} = \mathbf{C}$.
70. Let **A** and **B** be $n \times n$ nonsingular matrices. Show that **AB** is nonsingular.
71. Is it true that if **AB** is nonsingular, then both **A** and **B** must be nonsingular?

Problem Simon & Blume [1994], page 739, # 26, 28.

a) Prove that if the entries of **A** are all integers and if $|\mathbf{A}| = \pm 1$, then the entries of $\mathbf{A}^{-1}$ are also integers.

b) Use Theorem 5.3 (c) to show that if the entries of **A** and $\mathbf{A}^{-1}$ are all integers, then $|\mathbf{A}| = \pm 1$.

5.4 Cramer's Rule

The system of *n* linear equations with *n* variables can be solved by using the so-called Cramer's rule, which use the properties of the inverse and determinant.

Theorem 5.7 (Cramer's Rule) Given a system of n linear equations with n variables $\mathbf{Ax} = \mathbf{b}$, if $\mathbf{A}$ is nonsingular, then the solution to the system of linear equations are uniquely given by,

$$x_j = \frac{\left|\mathbf{a}_{\cdot 1} \;\; \cdots \;\; \mathbf{a}_{\cdot j-1} \;\; \mathbf{b} \;\; \mathbf{a}_{\cdot j+1} \;\; \cdots \;\; \mathbf{a}_{\cdot n}\right|}{|\mathbf{A}|}, \; j = 1,2,\ldots,n,$$

where the numerator of the right hand side of the equation is the determinant of matrix **A**, but with its column replaced by the right hand side vector **b**.

Proof Since $\mathbf{A}^{-1}$ exists, the solution is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, which by the calculation of by the adjoint matrix of **A** as stated in Theorem 5.5,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{|\mathbf{A}|}\begin{bmatrix} \sum_{i=1}^{n} b_i\Delta_{i1} \\ \sum_{i=1}^{n} b_i\Delta_{i2} \\ \vdots \\ \sum_{i=1}^{n} b_i\Delta_{in} \end{bmatrix}.$$

The theorem follows since each of the terms $\sum_{i=1}^{n} b_i\Delta_{ij}, \;\; j = 1,2,\ldots,n$, is the determinant of the matrix $\left[\mathbf{a}_{\cdot 1} \;\; \cdots \;\; \mathbf{a}_{\cdot j-1} \;\; \mathbf{b} \;\; \mathbf{a}_{\cdot j+1} \;\; \cdots \;\; \mathbf{a}_{\cdot n}\right]$ as expanded along the j^{th} column **b**. Then x_j is uniquely determined as given in the statement of the theorem.❑

Problem Fraleigh & Beauregard [1995], page 272, #34. Find the unique solution (assuming that it exists) of the system of equations represented by the augmented matrix

$$\left[\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & 3b_1 \\ a_2 & b_2 & c_2 & d_2 & 3b_2 \\ a_3 & b_3 & c_3 & d_3 & 3b_3 \\ a_4 & b_4 & c_4 & d_4 & 3b_4 \end{array}\right].$$