Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

$$2, 3, 4, 6$$

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2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$P_a = \sum_{s=1}^{k} \pi_s m_s X_{sa}$$
$$= E[mX_a]$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

Since $P_s = \pi_s m_s$; $m_s = \dfrac{\delta U'(C_s^*)}{U'(C_0^*)} = \dfrac{\delta C_0^*}{C_s^*}$

$$\therefore P_s = \pi_s \delta \frac{C_0^*}{C_s^*} \quad \#$$

3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E[m_{01} R_s]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d(C_s/C_0)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U(C_0) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

$$m_s = \frac{\delta U'(C_{1s})}{U'(C_0)} = \delta\left(\frac{C_1}{C_0}\right)^{\gamma-1}$$

FOC : $\pi_s \delta \left(\dfrac{C_s}{C_0}\right)^{\gamma-1} R_s = 1$

Differentiate ; $\pi_s \delta(\gamma-1)\left(\dfrac{C_s}{C_0}\right)^{\gamma-2} R_s \, d\left(\dfrac{C_s}{C_0}\right) + \pi_s \delta \left(\dfrac{C_s}{C_0}\right)^{\gamma-1} dR_s = 0$

$$(\gamma-1)\left(\frac{C_s}{C_0}\right)^{-1} R_s \, d\left(\frac{C_s}{C_0}\right) = -dR_s$$

$$\frac{R_s}{(C_s/C_0)} d\left(\frac{C_s}{C_0}\right) = \frac{-dR_s}{\gamma-1}$$

$$\frac{R_s}{(C_s/C_0)} \frac{d(C_s/C_0)}{dR_s} = -\frac{1}{1-\gamma}$$

Elasticity of intertemporal substitution ; $\varepsilon^I \equiv \dfrac{R_s}{C_s/C_0} \cdot \dfrac{d(C_s/C_0)}{dR_s} = \dfrac{1}{\gamma-1}$ ※

4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows $\rightarrow$ Price of risk-free $= \frac{1}{R_f} = 0.95$

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$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \rightarrow \text{dividend}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

a. Price of risk-free asset $= \frac{1}{R_f} = \frac{1}{1.05} = 0.9524$

$$P = \begin{bmatrix} 0.9524 \\ 6 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix} \rightarrow X^{-1} = \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}$$

Prices of the elementary securities :

$$p_1 = P'X^{-1}e_1 = [0.9524 \ \ 6]\begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.2476$$

$$p_2 = P'X^{-1}e_2 = [0.9524 \ \ 6]\begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.7048$$

the risk-neutral probabilities of the 2 states : $\hat{\pi}_s = p_s R_f$

state 1 ; $\hat{\pi}_1 = 0.2476 \times 1.05 = 0.26$

state 2 ; $\hat{\pi}_2 = 0.7048 \times 1.05 = 0.74$

b. $\pi_1 = \pi_2 = 0.5$

$$E[m] = \sum_{s=1}^{K} \pi_s m_s = \sum_{s=1}^{K} p_s = \frac{1}{R_f} = \frac{1}{1.05} = 0.9524$$

$$m_1 = \frac{p_1}{\pi_1} = \frac{0.2476}{0.5} = 0.4952$$

$$m_2 = \frac{p_2}{\pi_2} = \frac{0.7048}{0.5} = 1.4096$$

$$p_a = \sum_{s=1}^{2} p_s X_{sa}$$

$$\pi_1 m_1 + \pi_2 m_2 = \frac{1}{R_f}$$

$$0.5 m_1 + 0.5 m_2 = \frac{1}{1.05}$$

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6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

a. $E[m(R_i - R_f)] = 0$

$E[(a + bR_m)(R_i - R_f)] = 0$

$E[aR_i - aR_f + bR_mR_i - bR_mR_f] = 0$

$aE[R_i] - aR_f + bE[R_mR_i] - bR_fE[R_m] = 0$

note: $E[XY] = E[X] \cdot E[Y] + Cov(X, Y)$

so, $E[R_mR_i] = E[R_m] \cdot E[R_i] + Cov(R_m, R_i)$

$aE[R_i] - aR_f + bE[R_mR_i] - bR_fE[R_m] = 0$

$a(\cancel{R_f} + \beta_i\gamma) - \cancel{aR_f} + b[E[R_m]\,E[R_i] + \sigma_m^2] - bR_fE[R_m] = 0$

$a\beta_i\gamma + bE[R_m](R_f + \beta_i\gamma) + bCov(R_m, R_i) - bR_fE[R_m] = 0$

$a\beta_i\gamma + \cancel{bR_fE[R_m]} + b\beta_i\gamma E[R_m] + bCov(R_m, R_i) - \cancel{bR_fE[R_m]} = 0$

$\gamma\beta_i(a + bE[R_m]) = -bCov(R_m, R_i)$

$\gamma(a + bE[R_m]) = \dfrac{-bCov(R_m, R_i)}{\beta_i}$; $\beta_i = \dfrac{Cov(R_i, R_m)}{\sigma_m^2}$

$= -b\sigma_m^2$

$\therefore \gamma = \dfrac{-b\sigma_m^2}{a + bE[R_m]}$ ※

b. for risk-free asset:

$R_f = \dfrac{1}{E[m]}$

$R_f = \dfrac{1}{E[a + bR_m]}$

$R_f = \dfrac{1}{a + bE[R_m]}$

$a + bE[R_m] = \dfrac{1}{R_f}$

$\therefore a = \dfrac{1}{R_f} - bE[R_m]$ — 1

for market portfolio:

$1 = E[mR_m]$

$1 = E[(a + bR_m)(R_m)]$

$1 = E[aR_m + bR_m^2]$

$1 = aE[R_m] + bE[R_m^2]$ — 2

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Then, sub $a = \frac{1}{R_f} - bE[R_m]$ in ②

$$1 = \left(\frac{1}{R_f} - bE[R_m]\right)E[R_m] + bE[R_m^2]$$

$$1 = \frac{E[R_m]}{R_f} - bE[R_m]^2 + bE[R_m^2]$$

$$1 = \frac{E[R_m]}{R_f} - bE[R_m]^2 + b(\sigma_m^2 + E[R_m]^2)$$

$$1 = \frac{E[R_m]}{R_f} - bE[R_m]^2 + b\sigma_m^2 + bE[R_m]^2)$$

$$1 = \frac{E[R_m]}{R_f} + b\sigma_m^2$$

$$b\sigma_m^2 = 1 - \frac{E[R_m]}{R_f}$$

$$b = \frac{R_f - E[R_m]}{R_f\sigma_m^2} \text{ ※ } — ③$$

sub $b = \frac{R_f - E[R_m]}{R_f\sigma_m^2}$ in ①

$$a = \frac{1}{R_f} - \left(\frac{R_f - E[R_m]}{R_f\sigma_m^2}\right)E[R_m]$$

$$a = \frac{\sigma_m^2}{\sigma_m^2 R_f} - \frac{R_f - E[R_m]}{R_f\sigma_m^2}E[R_m]$$

$$a = \frac{\sigma_m^2 - E[R_m](R_f - E[R_m])}{R_f\sigma_m^2}$$

$$a = \frac{\sigma_m^2 - R_fE[R_m] + E[R_m]^2}{R_f\sigma_m^2} \text{ \#}$$

C. $$a + bE[R_m] = \frac{\sigma_m^2 - \cancel{R_fE[R_m]} + \cancel{E[R_m]^2}}{R_f\sigma_m^2} + \frac{\cancel{R_f} - \cancel{E[R_m]}}{R_f\sigma_m^2}\cancel{E[R_m]}$$

$$a + bE[R_m] = \frac{\sigma_m^2}{R_f\sigma_m^2} = \frac{1}{R_f} — ④$$

from question 1 ;

get $$\gamma = \frac{-b\sigma_m^2}{a + bE[R_m]} \text{ ※}$$

sub ③, ④

$$\gamma = \left(\frac{R_f - E[R_m]}{\cancel{R_f}\cancel{\sigma_m^2}} \cdot \cancel{\sigma_m^2}\right)\cancel{R_f}$$

$$\gamma = R_f - E[R_m] \text{ ※}$$