Optimization with Equality Constraints IV

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CW Ch.12

Outline

- Homogeneous Functions
- Least-Cost Combination of Inputs*

Homogeneous Functions

- A function is said to be homogeneous of degree r, if multiplication of each of its independent variables by a constant j will alter the value of the function by the proportion j^r, that is
$$f(jx_1, jx_2, \dots, jx_n) = j^r f(x_1, x_2, \dots, x_n).$$
- In general j can take any value. However, in economic applications, the constant j is usually taken to be positive.
- Example 1: Given the function $f(x, y, w) = \frac{x}{y} + \frac{2w}{3x}$, then
$$f(jx, jy, jw) = \frac{jx}{jy} + \frac{2(jw)}{3(jx)} = \frac{x}{y} + \frac{2w}{3x} = j^0 f(x, y, w).$$

  This function f is homogeneous of degree 0.

Homogeneous Functions

- Example 2: Given the function $g(x, y, w) = \frac{x^2}{y} + \frac{2w^2}{x}$, then
$$g(jx, jy, jw) = \frac{(jx)^2}{jy} + \frac{2(jw)^2}{jx} = \frac{jx^2}{y} + \frac{2jw^2}{x} = jg(x, y, w).$$
The function g is homogeneous of degree one.
- Example 3: Given the function $h(x, y, w) = 2x^2 + 3yw - w^2$, then
$$h(jx, jy, jw) = 2(jx)^2 + 3(jy)(jw) - (jw)^2 = j^2 h(x, y, w).$$
The function h is homogeneous of degree two.

Homogeneous Functions: Linear Homogeneity

- Homogeneous functions of degree one is widely used as production functions in economics.
- These are often referred to as ***linearly homogeneous functions***. Note that these functions need not be linear functions, but they are homogeneous of degree one.
- Now, given a production function

$$Q = f(K, L).$$

- The mathematical assumption of linear homogeneity would be equivalent to the economic assumption of constant returns to scale.
- Linear homogeneity means that raising all inputs j-fold will always raise the output exactly j-fold as well.

Homogeneous Functions: Linearly Homogeneous Production Functions

- **Property 1:** Given the linearly homogeneous production function $Q = f(K, L)$, the average physical product of labor (APP_L) and of capital (APP_K) can be expressed as functions of the capital-labor ratio, $k \equiv \frac{K}{L}$, alone.
- To show this, let $j = 1/L$. Since the production function is homogeneous of degree one, we have

$$f(jK, jL) = jQ \Rightarrow f\left(\frac{K}{L}, \frac{L}{L}\right) = \frac{Q}{L} \Rightarrow \frac{Q}{L} = f(k, 1).$$

- The function on the right side becomes a function of the capital-labor ratio alone, we have

$$APP_L \equiv \frac{Q}{L} = \phi(k).$$

- The expression for APP_K is then found to be

$$APP_K \equiv \frac{Q}{K} = \frac{Q}{L}\frac{L}{K} = \frac{\phi(k)}{k}.$$

Homogeneous Functions:
Linearly Homogeneous Production Functions

- **Property 1:** (Continued) Now, let's consider the average physical product of labor, which can be expressed as

  $$APP_L(K,L) = \frac{f(K,L)}{L}.$$

  If the production function is homogeneous of degree one, we can see that

  $$APP_L(jK,jL) = \frac{f(jK,jL)}{jL} = \frac{jf(K,L)}{jL} = APP_L(K,L).$$

- The average physical product of labor APP_L is homogeneous of degree *zero* in K and L. This is the same for APP_K.

  $$APP_K(jK,jL) = \frac{f(jK,jL)}{jK} = \frac{jf(K,L)}{jK} = APP_K(K,L).$$

- Linking to the previous slide, if the capital labor ratio $k \equiv K/L$ does not change, the APP_L and APP_K will not change either.

Homogeneous Functions: Linearly Homogeneous Production Functions

- **Property 2:** Given a linearly homogeneous production function $Q = f(K, L)$, the marginal physical products MPP_L and MPP_K can be expressed as functions of k alone.
- Before we figure out MPP_L and MPP_K, observe that

$$\frac{\partial k}{\partial K} = \frac{\partial}{\partial K}\left(\frac{K}{L}\right) = \frac{1}{L}, \text{ and } \frac{\partial k}{\partial L} = \frac{\partial}{\partial L}\left(\frac{K}{L}\right) = \frac{-K}{L^2}.$$

- Next, we can express the total product as

$$Q = L\phi(k).$$

- Therefore, the marginal physical products are

$$MPP_K \equiv \frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}[L\phi(k)] = L\frac{d\phi(k)}{dk}\frac{\partial k}{\partial K} = L\phi'(k)\frac{1}{L} = \phi'(k).$$

$$MPP_L \equiv \frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}[L\phi(k)] = \phi(k) + L\phi'(k)\frac{-K}{L^2} = \phi(k) - k\phi'(k).$$

These show that MPP_L and MPP_K are functions of k alone.

Example

- Consider the production function

$$Q = K^{1/2}L^{1/2}.$$

  Show that this function satisfies property 2.

Homogeneous Functions:
Linearly Homogeneous Production Functions

- **Property 3 (Euler's theorem):** If $Q = f(K, L)$ is linearly homogeneous, then

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = Q.$$

- Proof:

$$\begin{aligned} K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} &= K\phi'(k) + L[\phi(k) - k\phi'(k)] \\ &= K\phi'(k) + L\phi(k) - K\phi'(k) \\ &= L\phi(k) = Q. \end{aligned}$$

- Economically, this property means that under conditions of constant returns to scale, if each input factor is paid the amount of its marginal product, the total product will be exactly exhausted by the distributive shares of all the input factors.
- Equivalently, the pure economic profit will be zero.

Example

- Consider the production function

$$Q = K^{1/2}L^{1/2}.$$

  Show that this function satisfies property 3.

Homogeneous Functions: Cobb-Douglas Production Functions

- One specific production function widely used in economic analysis is the Cobb-Douglas production function:

  $$Q = AK^{\alpha}L^{1-\alpha},$$

  where A is a positive constant, and α is a positive fraction $0 < \alpha < 1$.
- However, first we consider the generalized version of this function:

  $$Q = AK^{\alpha}L^{\beta},$$

  where β is another positive fraction which may or may not be equal to $1 - \alpha$.
- Some of the major features of this function are:
  1) It is homogeneous of degree $\alpha + \beta$.
  2) In the special case of $\alpha + \beta = 1$, it is linearly homogeneous.
  3) Its isoquants are negatively sloped and convex for positive values of K and L.

Homogeneous Functions: Cobb-Douglas Production Functions

- Its homogeneity property can be easily seen from
$$A(jK)^{\alpha}(jL)^{\beta}= j^{\alpha+\beta}\left(AK^{\alpha}L^{\beta}\right) = j^{\alpha+\beta}Q.$$
That is the function is homogeneous of degree $(\alpha + \beta)$.
- From the above, if $(\alpha + \beta) = 1$, the function will be linearly homogeneous.
- For the property that its isoquants have negative slopes and strict convexity, let us apply the implicit-function rule:
$$\frac{dK}{dL} = -\frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}} = -\frac{\left(\frac{\beta Q}{L}\right)}{\left(\frac{\alpha Q}{K}\right)} = -\frac{\beta K}{\alpha L} < 0.$$
Next,
$$\frac{d^2K}{dL^2} = \frac{d}{dL}\left(-\frac{\beta K}{\alpha L}\right) = -\left(\frac{\alpha\beta L\left(\frac{dK}{dL}\right) - \alpha\beta K}{\alpha^2 L^2}\right) = -\frac{\beta}{\alpha}\frac{1}{L^2}\left(L\frac{dK}{dL} - K\right).$$
Since $\frac{dK}{dL} < 0$, we can say that $\frac{d^2K}{dL^2} > 0$. The isoquants are downward sloping and strictly convex for $K, L > 0$.

Homogeneous Functions: Cobb-Douglas Production Functions

- Let us now examine the $\alpha + \beta = 1$ case.
- First, the total product can be expressed as

$$Q = AK^{\alpha}L^{1-\alpha} = A\left(\frac{K}{L}\right)^{\alpha} L = LAk^{\alpha}.$$

Therefore,

$$APP_L = \frac{Q}{L} = Ak^{\alpha}$$

$$APP_K = \frac{Q}{K} = \frac{Q}{L}\frac{L}{K} = \frac{Ak^{\alpha}}{k} = Ak^{\alpha-1}.$$

- Second, differentiation of $Q = AK^{\alpha}L^{1-\alpha}$ yields the marginal products:

$$\frac{\partial Q}{\partial K} = \alpha AK^{\alpha-1}L^{-(\alpha-1)} = \alpha A\left(\frac{K}{L}\right)^{\alpha-1} = \alpha Ak^{\alpha-1}$$

$$\frac{\partial Q}{\partial L} = (1-\alpha)AK^{\alpha}L^{-\alpha} = (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} = (1-\alpha)Ak^{\alpha}.$$

These are functions of k alone.

Homogeneous Functions: Cobb-Douglas Production Functions

- Last, we can verify Euler's theorem as follows:

$$K\frac{\partial Q}{\partial K}+L\frac{\partial Q}{\partial L}=K\alpha Ak^{\alpha-1}+L(1-\alpha)Ak^{\alpha}$$

$$=LAk^{\alpha}\left(\frac{K\alpha}{Lk}+1-\alpha\right)$$

$$=LAk^{\alpha}(\alpha+1-\alpha)=LAk^{\alpha}=Q.$$

Here, if each input is assumed to be paid by the amount of its marginal product, the relative share of total product accruing to capital and labor will be:

$$\frac{K\left(\frac{\partial Q}{\partial K}\right)}{Q}=\frac{K\alpha Ak^{\alpha-1}}{LAk^{\alpha}}=\alpha,$$

$$\frac{L\left(\frac{\partial Q}{\partial L}\right)}{Q}=\frac{L(1-\alpha)Ak^{\alpha}}{LAk^{\alpha}}=(1-\alpha).$$

Homogeneous Functions: Extensions

- For a linearly homogeneous function

  $$y = f(x_1, x_2, \dots, x_n).$$

  We can multiply each variable by $\frac{1}{x_1}$ to get

  $$y = x_1 \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1} \dots, \frac{x_n}{x_1}\right) \quad \text{[because of homogeneity of degree 1].}$$

  The Euler's theorem can be extended to

  $$\sum_{i=1}^{n} x_i f_i = y,$$

  where the partial derivatives $f_i, i = 1, \dots, n$, are homogeneous of degree zero in the variables $x_i, i = 1, \dots, n$.

Assignment

- Exercise 12.6 No. 1 and 7