Exercise 3 (Part 3)

1. (a) Prove the statement:
   " For all integer n, if n is odd, then $5n+4$ is odd. "

   (b) Disprove the statement:
   " For all integers x and y, if $3x-5y$ is even, then x and y are both even. "

   (c) Prove or disprove the statement:
   " For all real numbers x and y, if x and y are rational, then x^y is rational. "

2. Show that "for all integer n, if $5n-1$ is even, then n is odd." by using
   a) a proof by contraposition,
   b) a proof by contradiction.

3. Prove or disprove that " $(n+1)^3 \geq 3^n$ for any positive integer n that is less than 5."

4. Use the **proof by cases** to show that " Prove that for all integers m and n, $m+n$ and $m-n$ are either both odd or both even."

   [Hint: Consider 4 cases of even and odd for m and n]

5. Let $P(n)$ be the statement " If $n > 1$, then $n^2 > n+1$ " with the domain of n consisting of all integers. Is the statement $P(1)$ true? Explain your answer.

6. Consider the statement: for any integer $n \geq 2$,

   $$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < \frac{2n-1}{n}.$$

   Suppose we want to prove the above statement by **mathematical induction**.

   (a) What is $P(n)$?
   (b) Write $P(2)$: Is $P(2)$ true?
   (c) Write $P(k)$:
   (d) Write $P(k+1)$:
   (e) Prove the above statement: $\sum_{j=1}^{n} j^{-2} < \frac{2n-1}{n}$ by using **mathematical induction**.

7. Use mathematical induction proof to show that

   $$2^n > n^2$$

   for an integer greater than 4.

8. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

9. (Optional) Use the method of constructive proof to show that:
   if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

10. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

11. (Optional) Show that at least four of any 22 days must fall on the same day of the week. [Hint: Use contradiction proof.]

12. (Optional) Show that these statements about the integer n are equivalent:
p_1: n is even. p_2: $n-1$ is odd. p_3: n^2 is even.

[Hint: We will show that these three statements are equivalent by showing that the conditional statements $p_1 \to p_2$, $p_2 \to p_3$, and $p_3 \to p_1$ are true.]

13. (Optional) A sequence $a_1, a_2, \ldots$ is defined recursively by

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$

Show that

$$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$