Chapter 8
Constrained Optimization: Equality

8.1 The Lagrangian Method: A maximization problem with a single equality constraint is given by

$$\max(\min)\ z = f(\mathbf{x})$$
$$st.\ g(\mathbf{x}) = c.$$

Write the Lagrange function,

$$\mathcal{L}(\mathbf{x},\lambda) = f(\mathbf{x}) - \lambda(g(\mathbf{x}) - c),$$

where λ is called the Lagrange multiplier.

First-Order Sufficient Conditions (Single Equality): The first-order sufficient conditions for the maximum point are given by the partial derivatives of the Lagrange function with respect to λ and each decision variable x_j are set to zero. That is,

$$\mathcal{L}_\lambda(\mathbf{x}^*,\lambda^*) = -g(\mathbf{x}^*) + c = 0$$
$$\mathcal{L}_j(\mathbf{x}^*,\lambda^*) = f_j(\mathbf{x}^*) - \lambda^* g_j(\mathbf{x}^*) = 0,\ j = 1,2,\ldots,n.$$

We have $n+1$ equations and $n+1$ unknowns which are the critical point $(\mathbf{x}^*,\lambda^*)$. The $n+1$ can be written equivalently in vector form as,

$$\nabla_{\left[\begin{smallmatrix}\lambda\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}(\mathbf{x}^*,\lambda^*) = \begin{bmatrix}\mathcal{L}_\lambda(\mathbf{x}^*,\lambda^*)\\ \nabla_\mathbf{x}\mathcal{L}(\mathbf{x}^*,\lambda^*)\end{bmatrix} = \begin{bmatrix}-g(\mathbf{x}^*)+c\\ \nabla f(\mathbf{x}^*) - \lambda^*\nabla g(\mathbf{x}^*)\end{bmatrix} = \mathbf{0}$$
.

Consequently, at the critical point $(\mathbf{x}^*,\lambda^*)$, the solution $\mathbf{x}^*$ is feasible and the gradient of f and the gradient of g point either in the same or opposite direction, i.e.,

$$\nabla f(\mathbf{x}^*) = \lambda^*\nabla g(\mathbf{x}^*).$$

In other words,

$$\frac{f_i(\mathbf{x}^*)}{f_j(\mathbf{x}^*)} = \frac{g_i(\mathbf{x}^*)}{g_j(\mathbf{x}^*)}.$$

This gives us the familiar micro-economic equilibrium condition: the ratio of the marginal products of any pair of input i, j is equal to their relative price.

Example: Consider the problem,

$$\begin{aligned} \boldsymbol{max}\, f(\mathbf{x}) &= 2x_1^2 + x_2^2 + 10x_1x_2 \\ st.\ g(\mathbf{x}) &= x_1 + 2x_2 = c. \end{aligned}$$

The first-order sufficient conditions are given by

$$\begin{aligned} \mathcal{L}_\lambda(\mathbf{x}^*, \lambda^*) &= -x_1^* - 2x_2^* + c = 0 \\ \mathcal{L}_1(\mathbf{x}^*, \lambda^*) &= 4x_1^* + 10x_2^* - \lambda^* = 0 \\ \mathcal{L}_2(\mathbf{x}^*, \lambda^*) &= 2x_2^* + 10x_1^* - 2\lambda^* = 0. \end{aligned}$$

The critical point is given by

$$\begin{aligned} x_1^* &= \frac{9c}{11} \\ x_2^* &= \frac{c}{11} \\ \lambda^* &= \frac{4c}{11}. \end{aligned}$$

HW Baldani, p. 241 #9.1 (a,b,d,e)—but replace inequality constraint by equality one.

Example: Consumer's Utility Maximization Problem.

$$\begin{aligned} &\max\ u(\mathbf{x}) \\ &st.\ \mathbf{p}^{\mathrm{T}}\mathbf{x} = I. \end{aligned}$$

The Lagrange function and the first-order sufficient conditions can be written as

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(\mathbf{p}^{\mathrm{T}}\mathbf{x} - I)$$

$$\mathcal{L}_\lambda\left(\mathbf{x}^*,\lambda^*\right) = -\mathbf{p}^\mathrm{T}\mathbf{x}^* + I_0 = 0$$
$$\mathcal{L}_j\left(\mathbf{x}^*,\lambda^*\right) = u_j\left(\mathbf{x}^*\right) - \lambda^* p_j = 0,\ j = 1,2,\dots,n.$$

Equivalently in matrix form,

$$\nabla_{\begin{bmatrix}\lambda\\ \mathbf{x}\end{bmatrix}}\mathcal{L}\left(\mathbf{x}^*,\lambda^*\right) = \begin{bmatrix} -\left(\mathbf{p}^\mathrm{T}\mathbf{x}^* - I\right) \\ \nabla u\left(\mathbf{x}^*\right) - \lambda^*\mathbf{p}\end{bmatrix} = \mathbf{0}\ .$$

We can solve for the optimal solution $\mathbf{x}^*$ in terms of prices $\mathbf{p}$ and income I. We can write $\mathbf{x}^* = \mathbf{x}_M\left(\mathbf{p}, I\right)$ and call it ***ordinary*** or ***Marshallian demand function***. The value of the utility at this optimal solution $u\left(\mathbf{x}_M\left(\mathbf{p}, I\right)\right) = v\left(\mathbf{p}, I\right)$ is ***called indirect utility function***.

From the first-order conditions, the marginal rate of substitution equal the relative price

$$\frac{u_i\left(\mathbf{x}^*\right)}{u_j\left(\mathbf{x}^*\right)} = \frac{p_i}{p_j},$$

and

$$\lambda^* = \frac{u_1\left(\mathbf{x}^*\right)}{p_1} = \cdots = \frac{u_n\left(\mathbf{x}^*\right)}{p_n},$$

which means the marginal utility from the last \$ spent must be the same for all goods and is equal to λ^*.

Example: Consumer's Expenditure Minimization problem.

$$\min\ \mathbf{p}^\mathrm{T}\mathbf{x}$$
$$st.\ u\left(\mathbf{x}\right) = u_0.$$

Write the Lagrange function:

$$\mathcal{L}\left(\mathbf{x},\lambda\right) = \mathbf{p}^\mathrm{T}\mathbf{x} - \lambda\left(u\left(\mathbf{x}\right) - u_0\right).$$

The first-order sufficient conditions are

$$\nabla_{\begin{bmatrix}\lambda\\ \mathbf{x}\end{bmatrix}}\mathcal{L}\left(\mathbf{x}^*,\lambda^*\right)=\begin{bmatrix}-\left(u\left(\mathbf{x}^*\right)-u_0\right)\\ \mathbf{p}-\lambda^*\nabla u\left(\mathbf{x}^*\right)\end{bmatrix}=\mathbf{0}\,.$$

Note that, as in the previous example, we also have the ratio

$$\frac{u_i\left(\mathbf{x}^*\right)}{u_j\left(\mathbf{x}^*\right)}=\frac{p_i}{p_j}\,.$$

We can solve for the optimal solution $\mathbf{x}^*$ in terms of $\mathbf{p}$ and u_0. Write $\mathbf{x}^*=\mathbf{x}_H\left(\mathbf{p},u_0\right)$ and call it the *Hicksian demand*. The value of the expenditure at this optimal solution is $e\left(\mathbf{p},u_0\right)=\mathbf{p}^{\mathrm{T}}\mathbf{x}_H\left(\mathbf{p},u_0\right)$, and called the expenditure function.

Note: If $u_0=u\left(\mathbf{x}_M\left(\mathbf{p},I\right)\right)=v\left(\mathbf{p},I\right)$, then

$$\mathbf{x}_H\left(\mathbf{p},u_0\right)=\mathbf{x}_M\left(\mathbf{p},e\left(\mathbf{p},u_0\right)\right).$$

8.2 Graphical Interpretation

For the case of two products, the consumer is allowed to choose the bundle of products only along the budget line in the $\left(x_1,x_2\right)$ plane, tracing an arc on the graph of the utility.

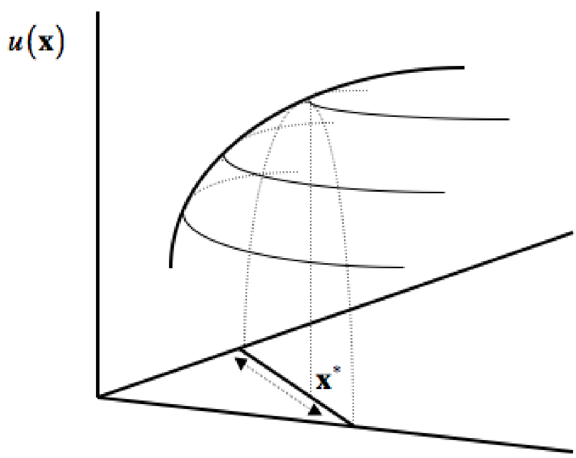

Figure 8.1 Graph of maximizing utility function subject to a linear budget constraint.

At the top of the arc where the utility is maximized at $\mathbf{x}^*$, the slope of the arc is zero and its curvature is concave. This observation yields respectively the first- and second-order sufficient conditions.

For the first-order sufficient conditions, along the direction that the budget is unchanged, for the infinitestimal changes in consumption **dx** such that

$$\begin{aligned} dg &= g_1\left(x_1^*,x_2^*\right)dx_1 + g_2\left(x_1^*,x_2^*\right)dx_2 = p_1dx_1 + p_2dx_2 = 0 \\ &= \nabla g\left(x_1^*,x_2^*\right)^{\mathrm{T}} \mathbf{dx} \\ &= p_1dx_1 + p_2dx_2 = 0, \end{aligned}$$

we also have

$$u_1\left(x_1^*,x_2^*\right)dx_1 + u_2\left(x_1^*,x_2^*\right)dx_2 = \nabla u\left(x_1^*,x_2^*\right)^{\mathrm{T}} \mathbf{dx} = 0 .$$

That is,

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2} = -\frac{u_1\left(x_1^*,x_2^*\right)}{u_2\left(x_1^*,x_2^*\right)} .$$

This last condition is obtained from the first-order condition of the Lagrangian method above. It says that at the optimal solution $\mathbf{x}^*$ the level sets of the constraint and the objective function are tangent. It also follows that the gradients of the constraint and the objective function either align in the same direction or the opposite direction.

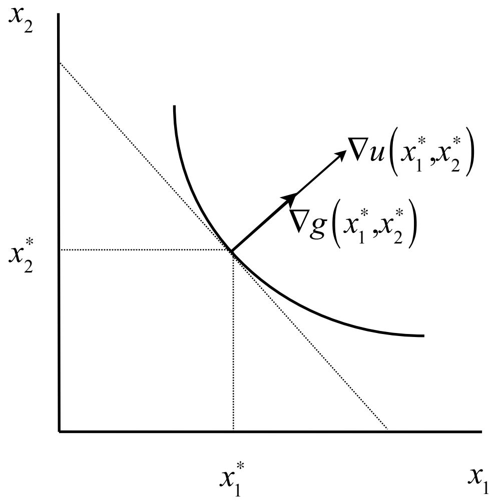

Figure 8.2 Gradient of constraint and objective function align in the same or opposite direction in the case of one constraint in 2-variable case.

That is, $\nabla u\left(x_1^*, x_2^*\right) = \lambda^* \nabla g\left(x_1^*, x_2^*\right)$ for some $\lambda^* \neq 0$. For the maximization of $f(\mathbf{x})$ with k equality constraints $\mathbf{g}(\mathbf{x}) = \mathbf{c}$ to be discussed in the next section, we will have $\nabla f\left(\mathbf{x}^*\right)^{\mathrm{T}} \mathbf{dx} = 0$ for the infinitestimal change **dx** such that $\nabla \mathbf{g}\left(\mathbf{x}^*\right)\mathbf{dx} = \mathbf{0}$, and it can be shown that for some $\boldsymbol{\lambda}^* \neq \mathbf{0}$, $\nabla f\left(\mathbf{x}^*\right) = \nabla \mathbf{g}\left(x_1^*, x_2^*\right)\boldsymbol{\lambda}^*$. See Sandaram [1996], pages 135-137.

For the second-order sufficient condition, the curvature of the graph of the objective function being concave at the optimal solution along this infinitestimal changes **dx**, where $\nabla \mathbf{g}\left(\mathbf{x}^*\right)\mathbf{dx} = \mathbf{0}$. That is, the Hessian of the objective function needs not be negative definite in all directions but just **dx** where $\nabla \mathbf{g}\left(\mathbf{x}^*\right)\mathbf{dx} = \mathbf{0}$. This is the test of the bordered Hessian to be discussed in Section 8.4.

8.3 Optimization with *k* Equality Constraints

$$\boldsymbol{max}\, f(\mathbf{x})$$
$$st.\ \mathbf{g}(\mathbf{x}) = \begin{bmatrix} g^1(\mathbf{x}) \\ g^2(\mathbf{x}) \\ \vdots \\ g^k(\mathbf{x}) \end{bmatrix} = \mathbf{c}.$$

where the number of constraints k is less than the number of decision variables n.

Write the Lagrange function,

$$\mathcal{L}(\mathbf{x},\boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda}^{\mathrm{T}}\left(\mathbf{g}(\mathbf{x}) - \mathbf{c}\right)$$
$$= f(\mathbf{x}) - \sum_{i=1}^{k} \lambda_i \left(g^i(\mathbf{x}) - c_i\right).$$

FOSC (*k* Equality Constraints):

$$\nabla_{\left[\begin{smallmatrix}\lambda \\ \mathbf{x}\end{smallmatrix}\right]} \mathcal{L}(\mathbf{x}^*,\lambda^*) = \begin{bmatrix} \nabla_\lambda \mathcal{L}(\mathbf{x}^*,\lambda^*) \\ \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*,\lambda^*) \end{bmatrix} = \begin{bmatrix} -\mathbf{g}(\mathbf{x}^*) + \mathbf{c} \\ \nabla f(\mathbf{x}^*) - \nabla \mathbf{g}(\mathbf{x}^*)^{\mathrm{T}} \lambda^* \end{bmatrix} = \mathbf{0}$$
$$= \begin{bmatrix} -\mathbf{g}(\mathbf{x}^*) + \mathbf{c} \\ \nabla f(\mathbf{x}^*) - \left[\nabla g^1(\mathbf{x}^*) \quad \nabla g^2(\mathbf{x}^*) \quad \cdots \quad \nabla g^k(\mathbf{x}^*)\right] \lambda^* \end{bmatrix}$$
$$= \begin{bmatrix} -g^1(\mathbf{x}^*) + c_1 \\ \vdots \\ -g^k(\mathbf{x}^*) + c_k \\ \nabla f(\mathbf{x}^*) - \sum_{i=1}^{k} \lambda_i^* \nabla g^i(\mathbf{x}^*) \end{bmatrix}.$$

We have $n+k$ equations and $n+k$ unknown being the critical point $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$. Consequently, at the critical point $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, the gradient of f can be written as a *linear combination* of the k gradients of the k constraints, i.e.,

$$\nabla f\left(\mathbf{x}^*\right) = \nabla \mathbf{g}\left(\mathbf{x}^*\right)^{\mathrm{T}} \boldsymbol{\lambda}^*$$
$$= \left[\nabla g^1\left(\mathbf{x}^*\right) \quad \nabla g^2\left(\mathbf{x}^*\right) \quad \cdots \quad \nabla g^k\left(\mathbf{x}^*\right)\right]\boldsymbol{\lambda}^*.$$
$$= \sum_{i=1}^{k} \lambda_i^* \nabla g^i\left(\mathbf{x}^*\right).$$

8.4 Second-Order Sufficient Conditions

With equality constraints, the curvature of the objective function needs to be concave only in the direction of change $\mathbf{dx}$ such that the value of the constraints do not change. That is, $df^2 = \mathbf{dx}^{\mathrm{T}}\mathbf{H}\left(\mathbf{x}^*\right)\mathbf{dx} < 0$ only for $\nabla \mathbf{g}\left(\mathbf{x}^*\right)\mathbf{dx} = \mathbf{0}$.

This condition is established by testing the ***leading principal minors*** of the ***bordered Hessian***.

8.4.1 Bordered Hessian for Single Equality Constraint The bordered Hessian is a $(n+1)\times(n+1)$ matrix as given by

$$\bar{\mathbf{H}}\left(\mathbf{x}^*,\lambda^*\right) = \nabla^2_{\left[\begin{smallmatrix}\lambda\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}\left(\mathbf{x}^*,\lambda^*\right) = \begin{bmatrix} 0 & -\nabla g\left(\mathbf{x}^*\right)^{\mathrm{T}} \\ -\nabla g\left(\mathbf{x}^*\right) & \nabla^2_{\mathbf{x}}\mathcal{L}\left(\mathbf{x}^*,\lambda^*\right) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\nabla g\left(\mathbf{x}^*\right)^{\mathrm{T}} \\ -\nabla g\left(\mathbf{x}^*\right) & \nabla^2 f\left(\mathbf{x}^*\right) - \lambda^*\nabla^2 g\left(\mathbf{x}^*\right) \end{bmatrix}.$$

8.4.2 Bordered Hessian for *k* Equality Constraints The bordered Hessian is a $(n+k)\times(n+k)$ matrix as given by

$$\bar{\mathbf{H}}\left(\mathbf{x}^*,\lambda^*\right) = \nabla^2_{\left[\begin{smallmatrix}\lambda\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}\left(\mathbf{x}^*,\lambda^*\right) = \begin{bmatrix} \mathbf{0} & -\nabla \mathbf{g}\left(\mathbf{x}^*\right) \\ -\nabla \mathbf{g}\left(\mathbf{x}^*\right)^{\mathrm{T}} & \nabla^2_{\mathbf{x}}\mathcal{L}\left(\mathbf{x}^*,\lambda^*\right) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{0} & -\nabla \mathbf{g}\left(\mathbf{x}^*\right) \\ -\nabla \mathbf{g}\left(\mathbf{x}^*\right)^{\mathrm{T}} & \nabla^2 f\left(\mathbf{x}^*\right) - \sum_{i=1}^{k}\lambda_i^*\nabla^2 g^i\left(\mathbf{x}^*\right) \end{bmatrix}.$$

8.4.3 Test of Bordered Matrix [Simon & Blume, Theorem 16.4, page 389] Given a bordered matrix

$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_{k\times n} \\ \mathbf{B}^{\mathrm{T}}_{n\times k} & \mathbf{A}_{n\times n} \end{bmatrix}_{(n+k)\times(n+k)}$$

where $\mathbf{A}$ is symmetric, the matrix $\mathbf{A}$ is ***negative definite*** subject to $\mathbf{Bd}=\mathbf{0}$ if $\left|\bar{\mathbf{H}}\right|$ has the same sign as $(-1)^n$ and the last $n-k$ leading principal minors alternate in sign.

The matrix $\mathbf{A}$ is ***positive definite*** subject to $\mathbf{Bd}=\mathbf{0}$ if $\left|\bar{\mathbf{H}}\right|$ and all the last $n-k$ leading principal minors have the same sign as $(-1)^k$.

In either case, $\bar{\mathbf{H}}$ is nonsingular.

Example: (Continued) For the problem,

$$\begin{aligned} \boldsymbol{max}\, f(\mathbf{x}) &= 2x_1^2 + x_2^2 + 10x_1x_2 \\ st.\ g(\mathbf{x}) &= x_1 + 2x_2 = c \end{aligned}$$

we had the FOSC,

$$\begin{aligned} \mathcal{L}_\lambda(\mathbf{x}^*,\lambda^*) &= -x_1^* - 2x_2^* + c = 0 \\ \mathcal{L}_1(\mathbf{x}^*,\lambda^*) &= 4x_1^* + 10x_2^* - \lambda^* = 0 \\ \mathcal{L}_2(\mathbf{x}^*,\lambda^*) &= 2x_2^* + 10x_1^* - 2\lambda^* = 0. \end{aligned}$$

The bordered Hessian is given by

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 4 & 10 \\ -2 & 10 & 2 \end{bmatrix}.$$

The matrix $\begin{bmatrix} \mathbf{4} & 10 \\ 10 & 2 \end{bmatrix}$ is neither positive nor negative definite. But the last $n-k=2-1=1$ leading principal minor, which is $\left|\bar{\mathbf{H}}\right| = 22 > 0$, being the same sign as $(-1)^n = (-1)^2 = 1 > 0$. Thus the bordered matrix $\bar{\mathbf{H}}$ is negative definite.

Example Utility Maximization Problem.

$$\boldsymbol{max}\, u(\mathbf{x}) = x_1 x_2 x_3$$
$$st.\ p_1 x_1 + p_2 x_2 + p_3 x_3 = I.$$

The Lagrange function is

$$\mathcal{L}(\mathbf{x}, \lambda) = x_1 x_2 x_3 - \lambda(p_1 x_1 + p_2 x_2 + p_3 x_3 - I),$$

and FOSC

$$\nabla_{\begin{bmatrix}\lambda\\ \mathbf{x}\end{bmatrix}} \mathcal{L}(\mathbf{x}^*, \lambda^*) = \begin{bmatrix} -(p_1 x_1^* + p_2 x_2^* + p_3 x_3^* - I) \\ x_2^* x_3^* - p_1 \lambda^* \\ x_1^* x_3^* - p_2 \lambda^* \\ x_1^* x_2^* - p_3 \lambda^* \end{bmatrix} = \mathbf{0}\ .$$

The bordered Hessian is

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & -p_1 & -p_2 & -p_3 \\ -p_1 & 0 & x_3^* & x_2^* \\ -p_2 & x_3^* & 0 & x_1^* \\ -p_3 & x_2^* & x_1^* & 0 \end{bmatrix}.$$

The last $n-1 = 3-1 = 2$ leading principal minors:

$$\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & 0 & x_3^* \\ -p_2 & x_3^* & 0 \end{vmatrix} = 2 p_1 p_2 x_3^* > 0$$

$$\left|\bar{\mathbf{H}}\right| = \qquad\qquad\qquad\qquad < 0$$

Thus $\bar{\mathbf{H}}$ is negative definite under the constraint and the critical point will be a maximum point.

Example The bordered Hessian of the utility maximization problem (Marshallian demand) is

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & -\mathbf{p}^{\mathrm{T}} \\ -\mathbf{p} & \nabla^2 u(\mathbf{x}^*) \end{bmatrix} = \mathbf{0}\ .$$

For expenditure minimization, (Hicksian demand)

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & -\nabla u(\mathbf{x}^*)^{\mathrm{T}} \\ -\nabla u(\mathbf{x}^*) & -\lambda^* \nabla^2 u(\mathbf{x}^*) \end{bmatrix}.$$

HW [As a continuation of Baldani, #9.1 (a,b)] p. 242 #9.2

8.5 Comparative Static Analysis: Sensitivity Analysis

Suppose that the constrained optimization contains a vector of parameters $\mathbf{d}_0 \in \boldsymbol{R}^p$. We write

$$\begin{aligned} &\boldsymbol{max}\, f(\mathbf{x};\mathbf{d}_0) \\ &st.\ \mathbf{g}(\mathbf{x};\mathbf{d}_0) = \mathbf{0}, \end{aligned}$$

where the parameters can be both in the objective function and the constraints. The right-hand-side of the constraints are now set to zero so that the parameters $\mathbf{d}_0$ here could also be the right-hand-side itself.

The Lagrange function is, for a given vector $\mathbf{d}$,

$$\mathcal{L}(\mathbf{x},\boldsymbol{\lambda};\mathbf{d}) = f(\mathbf{x};\mathbf{d}) - \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{g}(\mathbf{x};\mathbf{d}).$$

The first-order condition is thus a set of implicit functions:

$$\begin{aligned} \nabla_{\left[\begin{smallmatrix}\lambda \\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}(\mathbf{x}(\mathbf{d}),\lambda(\mathbf{d});\mathbf{d}) &= \begin{bmatrix} \nabla_{\lambda}\mathcal{L}(\mathbf{x}(\mathbf{d}),\lambda(\mathbf{d});\mathbf{d}) \\ \nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x}(\mathbf{d}),\lambda(\mathbf{d});\mathbf{d}) \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{g}(\mathbf{x}(\mathbf{d});\mathbf{d}) \\ \nabla_{\mathbf{x}} f(\mathbf{x}(\mathbf{d});\mathbf{d}) - \nabla_{\mathbf{x}}\mathbf{g}(\mathbf{x}(\mathbf{d});\mathbf{d})^{\mathrm{T}}\lambda(\mathbf{d}) \end{bmatrix} = \mathbf{0} \end{aligned}$$

If $(\mathbf{x}^*,\boldsymbol{\lambda}^*)$ is the optimal solution for the parameter vector $\mathbf{d}_0$, under the condition that

$$\nabla_{\left[\begin{smallmatrix}\lambda\\ \mathbf{x}\end{smallmatrix}\right]}\left[\nabla_{\left[\begin{smallmatrix}\lambda\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}\left(\mathbf{x}^*,\lambda^*;\mathbf{d}_0\right)\right]=\nabla^2_{\left[\begin{smallmatrix}\lambda\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}\left(\mathbf{x}^*,\lambda^*;\mathbf{d}_0\right)=\bar{\mathbf{H}}\left(\mathbf{x}^*,\lambda^*;\mathbf{d}_0\right)$$

is nonsingular, which is true if the second-order sufficient condition holds, then there is a differentiable function $\begin{bmatrix}\boldsymbol{\lambda}(\mathbf{d})\\ \mathbf{x}(\mathbf{d})\end{bmatrix}$ such that

a) $\nabla_{\left[\begin{smallmatrix}\boldsymbol{\lambda}\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}\left(\mathbf{x}(\mathbf{d}),\boldsymbol{\lambda}(\mathbf{d});\mathbf{d}\right)=\mathbf{0}$, for $\left\|\mathbf{d}-\mathbf{d}_0\right\|<\boldsymbol{\varepsilon}$,

b) $\begin{bmatrix}\boldsymbol{\lambda}(\mathbf{d}_0)\\ \mathbf{x}(\mathbf{d}_0)\end{bmatrix}=\begin{bmatrix}\boldsymbol{\lambda}^*\\ \mathbf{x}^*\end{bmatrix}$, and

c) the gradient

$$\nabla_{\mathbf{d}}\begin{bmatrix}\boldsymbol{\lambda}(\mathbf{d}_0)\\ \mathbf{x}(\mathbf{d}_0)\end{bmatrix}=-\left[\bar{\mathbf{H}}\left(\mathbf{x}^*,\boldsymbol{\lambda}^*;\mathbf{d}_0\right)\right]^{-1}\nabla_{\mathbf{d}}\left[\nabla_{\left[\begin{smallmatrix}\boldsymbol{\lambda}\\ \mathbf{x}\end{smallmatrix}\right]}\mathcal{L}\left(\mathbf{x}^*,\boldsymbol{\lambda}^*;\mathbf{d}_0\right)\right]$$

$$=-\begin{bmatrix}\mathbf{0} & -\nabla\mathbf{g}\left(\mathbf{x}^*;\mathbf{d}_0\right)\\ -\nabla\mathbf{g}\left(\mathbf{x}^*;\mathbf{d}_0\right)^{\mathrm{T}} & \nabla^2_{\mathbf{x}}L\left(\mathbf{x}^*,\boldsymbol{\lambda}^*;\mathbf{d}_0\right)\end{bmatrix}^{-1}\begin{bmatrix}-\nabla_{\mathbf{d}}\mathbf{g}\left(\mathbf{x}^*;\mathbf{d}_0\right)\\ \nabla_{\mathbf{d}}\left[\nabla_{\mathbf{x}}f\left(\mathbf{x}^*;\mathbf{d}_0\right)-\nabla_{\mathbf{x}}\mathbf{g}\left(\mathbf{x}^*;\mathbf{d}_0\right)^{\mathrm{T}}\boldsymbol{\lambda}^*\right]\end{bmatrix}.$$

We may write interchangeably $\nabla_{\mathbf{d}}\begin{bmatrix}\boldsymbol{\lambda}(\mathbf{d}_0)\\ \mathbf{x}(\mathbf{d}_0)\end{bmatrix}=\nabla_{\mathbf{d}}\begin{bmatrix}\boldsymbol{\lambda}^*\\ \mathbf{x}^*\end{bmatrix}$.

Example Minimizing expenditure under the single constraint of a given level of utility.

$$\boldsymbol{min}\ p_1x_1+p_2x_2$$
$$st.\ u(x_1,x_2)=u_0.$$

The Lagrange function is given by

$$\mathcal{L}(x_1,x_2;\bar{p}_1)=\bar{p}_1x_1+p_2x_2-\lambda\left(u(x_1,x_2)-u_0\right).$$

First-order condition:

$$\nabla_{\begin{bmatrix}\lambda\\ x_1\\ x_2\end{bmatrix}}\mathcal{L}\left(x_1^*,x_2^*;\overline{p}_1\right)=\begin{bmatrix}-\left(u\left(x_1^*,x_2^*\right)-u_0\right)\\ \overline{p}_1-\lambda^* u_1\left(x_1^*,x_2^*\right)\\ p_2-\lambda^* u_2\left(x_1^*,x_2^*\right)\end{bmatrix}=\mathbf{0}\,.$$

If the optimal is found by sufficient conditions, the bordered Hessian is nonsingular and we can apply the Implicit Function Theorem and have

$$\begin{bmatrix}\dfrac{\partial\lambda\left(\overline{p}_1\right)}{\partial p_1}\\ \dfrac{\partial x_1\left(\overline{p}_1\right)}{\partial p_1}\\ \dfrac{\partial x_2\left(\overline{p}_1\right)}{\partial p_1}\end{bmatrix}=-\begin{bmatrix}0 & -u_1\left(x_1^*,x_2^*\right) & -u_2\left(x_1^*,x_2^*\right)\\ -u_1\left(x_1^*,x_2^*\right) & -\lambda^* u_{11}\left(x_1^*,x_2^*\right) & -\lambda^* u_{12}\left(x_1^*,x_2^*\right)\\ -u_2\left(x_1^*,x_2^*\right) & -\lambda^* u_{21}\left(x_1^*,x_2^*\right) & -\lambda^* u_{22}\left(x_1^*,x_2^*\right)\end{bmatrix}^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

By Cramer's Rule and since $\boldsymbol{sgn}\left|\overline{\mathbf{H}}\right|=\left(-1\right)^k=-1$ with k being the number of constraint being one, we have

$$\frac{\partial x_1\left(\overline{p}_1\right)}{\partial p_1}=-\frac{\begin{vmatrix}0 & 0 & -u_2\left(x_1^*,x_2^*\right)\\ -u_1\left(x_1^*,x_2^*\right) & 1 & -\lambda^* u_{12}\left(x_1^*,x_2^*\right)\\ -u_2\left(x_1^*,x_2^*\right) & 0 & -\lambda^* u_{22}\left(x_1^*,x_2^*\right)\end{vmatrix}}{\left|\overline{\mathbf{H}}\right|}$$

$$=\frac{\left[u_2\left(x_1^*,x_1^*\right)\right]^2}{\left|\overline{\mathbf{H}}\right|}<0.$$

HW Do the sensitivity analysis for the case of

$$\boldsymbol{max}\, u\left(x_1,x_2\right)$$
$$st.\ p_1x_1+p_2x_2=I.$$

HW [As a continuation of Baldani, #9.1 (a,b)] p. 242 #9.3, 9.4. by using Implicit Function Theorem.