Answer: Exercise 2

Part I (*Derivative Definition, Power/Sum/Product/Quotient Rules*) .

1. Find the derivative $f'(x)$ of the following functions $f(x)$ by using the definition of the derivative:
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

   (a) $f(x) = 2x^2 + x + 1$
   Solution

$$\begin{aligned} f(x+h)-f(x) &= 2(x+h)^2 + (x+h) + 1 - (2x^2 + x + 1) \\ &= 2(x^2 + 2xh + h^2) + (x+h) + 1 - 2x^2 - x - 1 \\ &= 4xh + h^2 + h \end{aligned}$$

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4xh + h^2 + h}{h} = \lim_{h\to 0} 4x + h + 1 = 4x + 1.$$

   (b) $f(x) = \cos(x)$
   Hint: Use the identity: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, and the fact that $\lim_{h\to 0} \frac{\sin(h)}{h} = 1, \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.
   Solution

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$$

$$\begin{aligned} \frac{d}{dx}\cos(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h\to 0} \frac{[\cos(x)\cos(h) - \sin(x)\sin(h)] - \cos(x)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)[\cos(h)-1] - \sin(x)\sin(h)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)[\cos(h)-1]}{h} - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h} \\ &= \cos(x) \underbrace{\lim_{h\to 0} \frac{[\cos(h)-1]}{h}}_{=0} - \sin(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{=1} \\ &= -\sin(x). \end{aligned}$$

2. (a) $\frac{d}{dx}(\pi^{15}) = 0.$
   (b) $8x^7 + x^{-4} + x^{1/2}$

(c) $10x^9 + 9x^8 + 8x^7 - 6x^5 - 5x^4 - 4x^3 + \pi$

(d) $\frac{\left(x^2+x+1\right)(8x^7-4x^3)-\left(x^8-x^4+\pi\right)(2x+1)}{(x^2+x+1)^2}$

3. $x = 4$ and $x = -4$.

Part II (*Derivatives of trigonometric functions, Chain Rule, Implicit Differentiation*) .

1. (a) $3\cos(x) + \sec^2(x)$.

   (b) $\frac{[x+\csc(x)][-\cos(x)\csc^2(x)-\cot(x)\sin(x)]-\cos(x)\cot(x)[1-\cot(x)\csc(x)]}{[x+\csc(x)]^2}$

2. (a) $100(x^3-1)(x^2+1)^{99}2x + 3x^2(x^2+1)^{100}$

   (b) $\frac{-9x^2\sin^2(\cos(\sqrt{x^3-2}))\cos(\cos(\sqrt{x^3-2}))\sin(\sqrt{x^3-2})}{2\sqrt{x^3-2}}$.

3. (a) $\frac{dy}{dx} = \frac{1}{3y^2-2-3\sec(3y)\tan(3y)}$.

   (b) $\frac{dy}{dx} = \frac{3x^2(x-y)^2+2y}{2x}$

4. $$\left.\frac{dy}{dx}\right|_{(\pi/2,1)} = \frac{\cos(\pi/2)}{1-(\pi/2)\cos(\pi/2)} = \frac{0}{1-0} = 0.$$

5. The equation of the tangent line: $y = \frac{1}{2}x + (\frac{\pi}{4} - \frac{1}{2})$.

6. The equation of the normal line at $(-2, 2)$ is: $y = -\frac{3}{8}x + \frac{5}{4}$.

7. $(x, y) = (2, -2)$ and $(-2, 2)$.

8. $(x, y) = (1, 2)$ and $(-1, -2)$.

9. $(x, y) = (3, 6)$ and $(-3, -6)$.

10. (a) $= \frac{1}{y^2}\left(\frac{y^2-x^2}{y}\right) = -\frac{25}{y^3}$

    (b) $\frac{\sin(y)}{(\cos(y)-1)^3} = \frac{x+y}{(\cos(y)-1)^3}$