Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s,\omega_s,\forall_t} \; E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income ($y_t = 0\,\forall t$) and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

$$J(W_t, I_t, t) \equiv \max_{C_s, W_s, \forall_T} E_t\left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \delta^T \left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

at T_1 ; $J(W_T, T) = E_T[B(W_T, T)] = B(W_T, T)$

using dynamically programing backwards toward present

$$J(W_{T-1}, T-1) = \max_{C_{T-1}, (w_i, T-1)} E_{T-1}[U(C_{T-1}, T-1) + B(W_T, T)]$$

$$= \max_{C_{T-1}, (w_i, T-1)} U[C_{T-1}, T-1] + E_{T-1}[B(W_T, T)]$$

recall $S_{T-1} = W_{T-1} + y_{T-1} - C_{T-1}$

$R_{T-1} = R_{f,T-1} + \sum_{i=1}^{n} w_{i,T-1}(R_{i,T-1} - R_{f,T-1})$

FOC. $$\frac{\partial J}{\partial C_{T-1}} = \delta^{T-1}(1-\gamma)\frac{C_{T-1}^{-\gamma}}{(1-\gamma)} - E_{T-1}\left[\delta^T \cancel{(1-\gamma)}\frac{W_T^{-\gamma}}{\cancel{(1-\gamma)}} \cdot R_{T-1}\right] = 0 \quad \text{①}$$

$$\frac{\partial J}{\partial w_{i,T-1}} = E_{T-1}\left[\delta^T \cancel{(1-\gamma)} \frac{W_T^{-\gamma}}{\cancel{(1-\gamma)}}(R_{i,T-1} - R_{f,T-1})\right] = 0 \quad \text{②}$$

rewritten ① using ②

$$\delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}\left[\delta^T W_T^{-\gamma}\left(R_{f,T-1} + \sum_{i=1}^{n} w_{i,T-1}(R_{i,T-1} - R_{f,T-1})\right)\right]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = R_{f,T-1}\, E_{T-1}[\delta^T W_T^{-\gamma}]$$

$\therefore C^*_{T-1}$ gives an expression of right-now than the futu

Score....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J_W = U_c \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left(\frac{dW_T}{dW_{T-1}} \right) \right]$$

$$= U_c \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left(\frac{\partial W_T}{\partial W_{T-1}} + \sum_{i=1}^{n} \frac{\partial W_T}{\partial W^*_{i,T-1}} \frac{\partial W^*_{i,T-1}}{\partial W_{T-1}} + \frac{\partial W_T}{\partial C^*_{T-1}} \frac{\partial C^*_{T-1}}{\partial W_{T-1}} \right) \right]$$

$$= U_c \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left(\sum_{i=1}^{n} [R_{i,T-1} - R_{f,T-1}]\, S_{T-1} \frac{\partial W^*_{i,T-1}}{\partial W_{T-1}} + R_{T-1}\left(1 - \frac{\partial C^*_{T-1}}{\partial W_{T-1}} \right) \right) \right]$$

FOC $\quad E_{T-1}[B_{W_T}(R_{i,T-1} - R_{f,T-1})] = 0$

$$U_c = E_{T-1}(B_{W_T} R_{T-1})$$

Simplified $\quad J_W = E_{T-1}[B_{W_T} R_{T-1}]$

$$J_W(W_{T-1}, T-1) = U_c(C^*_{T-1}, T-1)$$

$$J(W_{T-1}, T-1) = \delta^{T-1} \ln[C^*_{T-1}] + \delta^{T} E_{T-1}[\ln[R^*_{T-1}(W_{T-1} - C^*_{T-1})]]$$

$$= \delta^{T-1}(-\ln[1+\delta] + \ln[W_{T-1}]) +$$

$$\delta^{T}\left(E_{T-1}[\ln(R^*_{T-1})] + \ln\left[\frac{\delta}{1+\delta}\right] + \ln(W_{T-1}) \right)$$

$$= \delta^{T-1}[(1+\delta)\ln[W_{T-1}] + H_{T-1}]$$

where $H_{T-1} \equiv -\ln[1+\delta] + \delta \ln\left[\frac{\delta}{1+\delta}\right] + \delta E_{T-1}[\ln[R^*_{T-1}]]$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$J(W_{T-2}, T-2) = \max_{C_{T-2}, (w_i, T-2)} U[C_{T-2}, T-2] + E_{T-2}[J(W_{T-1}, T-1)]$$

FOC

$$\frac{\partial J}{\partial C_{T-1}} = \delta^{T-2}(1-\gamma)\frac{C_{T-2}^{-\gamma}}{(1-\gamma)} - E_{T-2}\left[\delta^{T}\cancel{(1-\gamma)}\frac{W_T^{-\gamma}}{\cancel{(1-\gamma)}} \cdot R_{T-2}\right] = 0$$

$$\frac{\partial J}{\partial w_{i,T-2}} = E_{T-2}\left[\delta^{T}\cancel{(1-\gamma)}\frac{W_T^{-\gamma}}{\cancel{(1-\gamma)}}\right](R_{i,T-2} - R_{f,T-2}) = 0$$

$$\delta^{T-2} C_{T-2}^{-\gamma} = R_{f,T-2}\, E_{T-2}(\delta^{T} W_T^{*\gamma})$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = \max_{c_{T-2}, \{w_i, T-2\}} U(c_{T-2}, T-2) + E_{T-2}\left[J(W_{T-1}, T-1)\right]$$

$$= \max_{c_{T-2}, \{w_i, T-2\}} \delta^{T-2} \ln[c_{T-2}] + \delta^{T-1} E_{T-2}\left[(1+\delta)\ln[W_{T-1}] + H_{T-1}\right]$$