EE320 Chapter 8
Optimization without Constraint: More-Than-One-Independent-Variable Case

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1 Review

Recall from one-independent variable optimizations without constraints:

$$\begin{aligned} y &= f(x) \\ dy &= f'(x)dx \end{aligned}$$

FOC for an optimum <u>necessary condition</u>

$\frac{dy}{dx} = f'(x) = 0$ or first-order differential condition for any dx $\neq 0$ and $dy = 0$

SOC to check <u>sufficient condition</u>

$$\begin{aligned} \frac{d^2y}{dx^2} = f''(x) &< 0 \text{ maximum} \\ f''(x) &> 0 \text{ minimum} \\ f''(x) &< 0 \text{ indetermined (max,min,inflection point)} \\ \Rightarrow d^2y &= d(dy) = d(f'(x)dx) = (df'(x))dx = f''(x)dx \end{aligned}$$

2 Two-choice-variable optimization

Suppose that we have a function with 2 choice variables: $z = f(x, y)$

First-order-necessary condition is :

$$\begin{aligned} dz &= f_x dx + f_y dy = 0 \\ \therefore dz &= 0 \text{ when } f_x = f_y = 0 \end{aligned}$$

Theorem: A differentiable function $z = f(x, y)$ can only have a maximum or minimum at an interior point (x_0, y_0) if it is a stationary point. That is , if the point $(x.y) = (x^*, y^*)$ satisfies the two FOC equations:

$$f_x(x_0^*, y_0^*) = 0$$
$$f_y(x_0^*, y_0^*) = 0$$

ex. $z = f(x, y) = 10x + 10y + xy - x^2 - y^2$

FOC: $\frac{\partial z}{\partial x} =$

$\frac{\partial z}{\partial y} =$

Next, find whether those points give max, min or saddle point.

$\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ are necessary but no yet sufficient for max/min.

3 Second-Order Conditions

$$\begin{aligned}
d^2z &= d(dz) \qquad \text{suppose } z = f(x,y) \\
&= \tfrac{\partial}{\partial x} dz dx + \tfrac{\partial}{\partial y} dz dy \\
&= \tfrac{\partial}{\partial x}(f_x dx + f_y dy)dx + \tfrac{\partial}{\partial y}(f_x dx + f_y dy)dy \\
&= (f_{xx}dx + f_{xy}dy)dx + (f_{yx}dx + f_{yy}dy)dy \\
&= f_{xx}dx^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}dy^2 \\
d^2z &= f_{xx}dx^2 + 2f_{xy}dydx + f_{yy}dy^2 \\
&= \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}
\end{aligned}$$

$\downarrow$

Hessian matrix or principal minor

$$
\begin{aligned}
\underline{\text{Note}} \qquad q &= au^2 + 2huv + bv^2 \\
&= au^2 + 2huv + \tfrac{h^2}{a}v^2 + bv^2 - \tfrac{h^2}{a}v^2 \\
&= a(u^2 + \tfrac{2h}{a}uv + \tfrac{h^2}{a^2}v^2) - (b - \tfrac{h^2}{a})v^2 \\
&= a(u + \tfrac{h}{a}v)^2 + (\tfrac{ab-h^2}{a})v^2
\end{aligned}
$$

$q>0$ when $a>0$ and $ab - h^2>0$

$q<0$ when $a<0$ and $ab - h^2>0$

$$
\begin{array}{lllcllll}
q &=& d^2z &,& u &=& dx \\
a &=& f_{xx} &,& y &=& dy \\
b &=& f_{yy} &,& h &=& f_{xy}
\end{array}
$$

First leading principal minor $|H_1| = f_{xx}$

Second leading principal minor $|H_2| = f_{xx}f_{yy} - f_{xy}^2$

<u>Second-order sufficient condition</u>

Given that FOC is satisfied

1. maximum when $d^2z<0$ iff $f_{xx}<0$

$f_{yy}<0$

$f_{xx}f_{yy}>(f_{xy})^2$ or $|H_2|>0$

2. minimum when $d^2z>0$ iff $f_{xx}>0$

$f_{yy}>0$

$f_{xx}f_{yy}>(f_{xy})^2$ or $|H_2|>0$

3. otherwise, saddle point

<u>ex.</u> $f(x,y) = 10x + 10y + xy - x^2 - y^2$

$$\text{FOC:} \quad \begin{aligned} f_x &= \\ f_y &= \end{aligned}$$

$$\text{SOC:} \quad \begin{aligned} f_{xx} &= \\ f_{xy} &= \\ f_{yx} &= \\ f_{yy} &= \end{aligned}$$

H =

$|H_1| =$

$|H_2| =$

<u>ex.</u> $f(x,y) = -2x^2 - 2xy - 2y^2 + 32x + 42y - 158$

$$\left.\begin{aligned} x^* &= 5 \\ y^* &= 8 \\ |H_1| &< 0 \\ |H_2| &= 12 \quad > 0 \end{aligned}\right\} \text{max}$$

$$\underline{\text{ex.}}\ \ f(x,y) = -x^2 + xy - y^2 + 3x$$

$$\begin{aligned} (2,1) \quad z^* &= 3 \\ |H_1| &< 0 \\ |H_2| &> 0 \end{aligned}$$

4 Concavity and Convexity: 2-independent-variable-case

Consider a function with 2 choice variable: $f(x_1, x_2)$

Definition the function is (strictly) concave iff for any pair of distinct points M & N on its graph or surface, line segment MN lies either below or entirely below

Definition the function is (strictly) convex iff for any pair of distinct points M & N on its graph or surface, line segment MN lies either above or entirely above

$$\text{or} \qquad \overbrace{\alpha f(x) + (1-\alpha) f(x')}^{\text{“Height of MN”}} < \overbrace{f(\alpha x + (1-\alpha)x')}^{\text{“Height of arc”}} \Rightarrow \text{strictly concave}$$

$$\alpha f(x) + (1-\alpha) f(x') > f(\alpha x + (1-\alpha)x') \Rightarrow \text{strictly convex}$$

If we use derivative conditions;

Consider

$z = f(x_1, x_2)$ - twice continuously differentiable $\rightarrow d^2z$ is defined.

$z = f(x_1, x_2)$ is strictly concave iff d^2z is everywhere definite negative or $d^2z<0$

$\Rightarrow z^*$ is max.

$z = f(x_1, x_2)$ is strictly convex iff d^2z is everywhere definite positive or $d^2z>0$

$\Rightarrow z^*$ is min.

ex. $z = x_1^2 - x_2^2$

ex. $z = x_1^2 + x_2^2$

ex. $z = 2x^2 - xy + y^2$ (convex)

Application: Duopoly

There are 2 firms with identical cost $TC_i = cQ_i$, where $i = 1, 2$ and market demand $P = a - bQ$, $Q = Q_1 + Q_2$ Find Q_1, Q_2 that maximizes each firm's profit.

$$
\begin{aligned}
\text{Firm 1: } \Pi_1 &= P(Q) \cdot Q_1 - cQ_1 \\
&= [a - b(Q_1 + Q_2)]Q_1 - cQ_1 \\
\Pi_1 &= (a - c)Q_1 - bQ_1^2 - bQ_1Q_2 \\
\max_{Q_1} \Pi_1 &= (a - c)Q_1 - bQ_1^2 - bQ_1Q_2 \\
\underline{\text{FOC}} \quad \frac{\partial \Pi}{\partial Q_1} &=
\end{aligned}
$$

$\Rightarrow Q_1^* =$ firm's best response function

$$
\begin{aligned}
\text{Firm 2: } \Pi_2 &= P(Q) \cdot Q_2 - cQ_2 \\
\underline{\text{FOC}} \quad Q_2^* &=
\end{aligned}
$$

firm's best response function

ex. Given the following information, market demand $P = 150 - Q$, $Q = Q_1 + Q_2$, $P = P_1 = P_2$ and $TC_1 = 60Q_1$, $TC_2 = 60Q_2$. Find optimal quantity that each firm should produce. Also check for their SOC.

Application: Duopoly-extension

1. Cournot: Move at the same time

2. Stackelberg: One seller can move first

Suppose that there are two firms and they face the market demand $P = 30 - Q$, where $Q = q_1 + q_2$. Their marginal cost structures are the same: $MC_1 = MC_2 = 3$. Consider the following scenarios:

1) Both move simultaneously
2) Firm 1 leads/moves first
3) Firm 2 leads/moves first
4) Both try to lead

Application: Multiproduct firm

1. Suppose a firm produce two goods where both of them are selling into perfectly competitive markets. Given all the following information of market structure: $P_1 = 6$, $P_2 = 9$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$. Find Q_1 and Q_2 that maximize its profit Π

$$\begin{aligned}
\Pi(Q_1, Q_2) &= TR_1 + TR_2 - TC \\
\max_{Q_1,Q_2} \Pi(Q_1, Q_2) &= 6Q_1 + 9Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)
\end{aligned}$$

Suppose instead that this particular firm operate as a monopoly for both goods. Consider all the following information: $P_1 = 35-Q_1$, $P_2 = 33-Q_2$ and $TC = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$. Find Q_1 and Q_2 that maximize Π

$$\begin{aligned}
\Pi(Q_1, Q_2) &= TR_1 + TR_2 - TC \\
\max_{Q_1,Q_2} \Pi(Q_1, Q_2) &= (35 - Q_1)Q_1 - (33 - Q_2)Q_2 - (2Q_1^2 + Q_1Q_2 + 2Q_2^2)
\end{aligned}$$

Application: Multiplant firm

Consider a firm that operate in perfectly competitive market with $P = 25$ and has two factories which each has the following cost structures $TC_1 = 2Q_1^2 + 5Q_1 + 10$ and $TC_2 = 2Q_2^2 + 3Q_2 + 15$. Find Q_1 and Q_2 that maximize firm's profit: $\Pi(Q_1, Q_2) = TR_1 - TC_1 - TC_2$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2) = 25(Q_1 + Q_2) - (2Q_1^2 + 5Q_1 + 10) - (2Q_2^2 + 3Q_2 + 15)$$

Application: Multimarket Monopoly or Price discrimination

Suppose that a firm has certain market power over 2 goods. The total revenue for goods 1 and 2 is $R = R_1(Q_1) + R_2(Q_2)$ and $C = C(Q)$ where $Q = Q_1 + Q_2$. Find FONC and SOSC for maximum profit Π

$$\Pi = R_1(Q_1) - R_2(Q_2) - C(Q)$$

$$\max_{Q_1, Q_2} \Pi(Q_1, Q_2)$$

<u>FONC</u>

$$\frac{\partial \Pi}{\partial Q_1} =$$

$$\frac{\partial \Pi}{\partial Q_2} =$$

Solve for (Q_1^*, Q_2^*): Firms need to produce until: $MR_1 = MR_2 = MC$

<u>SOSC</u>

$$\Pi_{11} =$$

$$\Pi_{22} =$$

$$\Pi_{12} = \Pi_{21} =$$

SOSC is satisfied when
1. $R_1'' - C'' < 0$
2. $R_2'' - C'' < 0$
3. $\Pi_{11}\Pi_{22} - \Pi_{12}^2 > 0$

<u>ex.</u> $P_1 = 22 - 2Q_1, P_2 = 10 - 0.5Q_2$ and$TC = 2Q + 5$. Find Q_1, Q_2, P_1 and P_2 that maximize firm's profit.

$$\max_{Q_1,Q_2} \Pi(Q_1, Q_2) = (22 - 2Q_1)Q_1 + (10 - 0.5Q_2) - (2Q + 5)$$

Application: Input decision of a firm

Let $Q = f(K, L) = 5K^{0.5}L^{0.25},\ P = 4,\ w = 10,\ r = 5$. Find $K^*,\ L^*$ max profit.

$$\Pi = P \cdot Q - TC$$

$$= 4 \cdot (5K^{0.5}L^{0.25}) - (10K + 5L)$$

$$\max_{K,L} \Pi(K, L)$$

<u>FONC</u> :

$$\frac{\partial \Pi}{\partial K} = 10K^{-0.5}L^{0.25} - 5 = 0 \quad \Rightarrow \quad 2K^{-0.5}L^{0.25} - 1 = 0$$

$$\frac{\partial \Pi}{\partial L} = 5K^{-0.5}L^{-0.75} - 10 = 0 \quad \Rightarrow \quad K^{0.5}L^{-0.75} - 2 = 0$$

$$K^* = 4$$
$$L^* = 1$$

$$\begin{aligned}
\underline{SOSC} \quad \Pi_{KK} &= -5K^{-1.5}L^{0.25} &\Rightarrow\quad \Pi_{KK}(K^*=4, L^*=1) = -\tfrac{5}{8}<0 \\
\Pi_{LL} &= -3.75K^{0.5}L^{-1.75} &\Rightarrow\quad \Pi_{LL}(K^*=4, L^*=1) = -\tfrac{15}{2}<0 \\
\Pi_{KL} &= 2.5K^{-0.5}L^{-0.75} &\Rightarrow\quad \Pi_{KL}(K^*=4, L^*=1) = -\tfrac{5}{4}
\end{aligned}$$

$$H = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} \\ \frac{5}{4} & -\frac{15}{2} \end{bmatrix} \qquad |H_1|<0\ ,\ |H_2|>0 \therefore \Pi(4,1) \text{ is max.}$$

5 Comparative-Static Aspects of Optimization

We can use partial differentiation to study how a change in exogenous variables affect the equilibrium outcome in the model.

ex. Consider a perfectly competitive markets. P_1 and P_2 are exogenous variables. The total revenue of the firm is $R = P_1Q_1 + P_2Q_2$, and its cost is $C = 2Q_1^2 + Q_1Q_2 + Q_2^2$. Find how equilibrium quantity of goods 1 and 2 will be affected after a change in market prices P_1 and P_2.

$$\max_{Q_1,Q_2} \Pi(Q_1, Q_2) = P_1Q_1 + P_2Q_2 - (2Q_1^2 + Q_1Q_2 + Q_2^2)$$

ex. Consider $R = P \cdot Q(K, L)$ where P, w, and r are exogenous. Find $\frac{\partial L^*}{\partial P}, \frac{\partial K^*}{\partial P}$. Given, $C = wL + rK$

$$\max_{K,L} \Pi(K, L) = P \cdot Q(K, L) - (wL + rK)$$

FONC $\Pi_K =$

$\Pi_L =$

$$\Rightarrow K^* = K^*(P, w, r)\ ,\ L^* = L^*(P, w, r)$$

If P changes, while w and r remain constant; what will happen to K^*, L^*?

$$K^* = K^*(P) = K^*(P, \bar{w}, \bar{r})$$

$$L^* = L^*(P) = L^*(P, \bar{w}, \bar{r})$$

For FONC:

$$\Pi_K = P \cdot Q_K(K^*(P), L^*(P)) - r = 0$$

$$\Pi_L = P \cdot Q_L(K^*(P), L^*(P)) - w = 0$$

$\frac{d\Pi_K}{dP} =$

$=$

$=$

$\therefore \frac{d\Pi_K}{dP} =$

$\frac{d\Pi_L}{dP} =$

$$\begin{bmatrix} P \cdot Q_{KK} & P \cdot Q_{LK} \\ P \cdot Q_{KL} & P \cdot Q_{LL} \end{bmatrix} \begin{bmatrix} \frac{dK^*}{dP} \\ \frac{dL^*}{dP} \end{bmatrix} = \begin{bmatrix} -Q_K \\ -Q_L \end{bmatrix}$$

$\frac{dK^*}{dP} =$

$\frac{dL^*}{dP} =$

$\therefore$ L and K are complement

6 Multivariable Optimization

Let a function with 3 choice variables:

$$z = f(x_1, x_2, x_3)$$
$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

<u>FONC</u>: $dz = 0 \Leftrightarrow f_1 = f_2 = f_3 = 0$

<u>SOSC</u>:

$$\begin{aligned} d^2z &= \frac{\partial}{\partial x_1}(dz)dx_1 + \frac{\partial}{\partial x_2}(dz)dx_2 + \frac{\partial}{\partial x_3}(dz)dx_3 \\ &= \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix} \underbrace{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}}_{\text{Hessian Matrix}} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \end{aligned}$$

Given any Hessian matrix: $H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$

$$
\left.\begin{aligned}
|H_1| &= f_{11} \\
|H_2| &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \\
|H_3| &= \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}
\end{aligned}\right\}
$$

$d^2z < 0$ (max)	$d^2z > 0$ (min)
$\|H_1\|<0$	$\|H_1\|>0$
$\|H_2\|>0$	$\|H_2\|>0$
$\|H_3\|<0$	$\|H_3\|>0$
"negative definite"	"positive definite"
$\Rightarrow$ all principal minor alternate its sign	$\Rightarrow$ all principal minor must be positive

<u>Summary</u>

	max	min
FONC	$f_1 = f_2 = \cdots = f_n = 0$	$f_1 = f_2 = \cdots = f_n = 0$
SOSC	$\|H_1\| < 0$, $\|H_2\| > 0$ $\|H_3\| < 0, ...$ or $(-1)^i\|H_i\| > 0$ for $i = 1, 2, \ldots, n$	$\|H_1\|>0$, $\|H_2\| > 0$, $\|H_3\| > 0, ...$ or $\|H_i\| > 0$ for $i = 1, 2, \ldots, n$

ex. $y = x_1^2 + 6x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3$

$\frac{\partial y}{\partial x_1} =$

$\frac{\partial y}{\partial x_2} =$

$\frac{\partial y}{\partial x_3} =$

$$\left.\begin{array}{ll} f_{11} & = \\ f_{22} & = \\ f_{33} & = \\ f_{12} & = \\ f_{13} & = \\ f_{21} & = \\ f_{23} & = \\ f_{31} & = \\ f_{32} & = \end{array}\right\} H = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

$|H_3| =$

$\therefore d^2y > 0$ positive definite (min)

Application: Multimarket Monopoly

$$\text{Let}\quad \begin{aligned} P_1 &= 63 - 4Q_1 \\ P_2 &= 105 - 5Q_2 \\ P_3 &= 75 - Q_3 \\ TC &= 15Q + 20 \end{aligned}$$

Find Q_1 and P_1 for $i = 1, 2, 3$ that max Π

$$\max_{Q_1,Q_2,Q_3} \Pi(Q_1, Q_2, Q_3)$$

$$= (63-4Q_1)Q_1+(105-5Q_2)Q_2+(75-Q_3)Q_3-[15(Q_1+Q_2+Q_3)+20]$$

$$\underline{FONC} \qquad \begin{aligned} \Pi_1 &= 48 - 8Q_1 &\Rightarrow\quad Q_1^* = 6 \\ \Pi_2 &= 90 - 10Q_2 &\Rightarrow\quad Q_2^* = 9 \\ \Pi_3 &= 60 - 20Q_3 &\Rightarrow\quad Q_3^* = 30 \end{aligned}$$

$$\underline{SOSC} \qquad \left.\begin{aligned} \Pi_{11} &= -8 \\ \Pi_{22} &= -10 \\ \Pi_{33} &= -2 \\ \Pi_{12} &= \Pi_{13} = 0 \\ \Pi_{21} &= \Pi_{23} = 0 \\ \Pi_{31} &= \Pi_{33} = 0 \end{aligned}\right\} H = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$|H_1| = -8 < 0$

$|H_2| = 80 > 0$

$|H_3| = -160 > 0$

$\therefore\ d^2z < 0$ negative definite

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