Chapter 11
Inequality Constrained Optimization: Applications

11.1 Utility Maximization with Two Goods The maximization of utility under the condition that the expenditure is less than or equal to a given budget with nonnegative consumption constraints is given by

$$\begin{aligned} &\max Z = U(x,y) \\ &\text{st.}\, p_x x + p_y y \le I \\ &\qquad\quad x, y \ge 0. \end{aligned}$$

The K-T Lagrange function is given by

$$\hat{\mathcal{L}}(x,y,\mu) = U(x,y) - \mu(p_x x + p_y y - I)$$

First-order K-T Conditions are

$$\begin{aligned} \hat{\mathcal{L}}_x(x^*,y^*,\mu^*) &= U_x(x^*,y^*) - \mu^* p_x \le 0 \\ \hat{\mathcal{L}}_y(x^*,y^*,\mu^*) &= U_y(x^*,y^*) - \mu^* p_y \le 0 \\ x^*\hat{\mathcal{L}}_x(x^*,y^*,\mu^*) &= x^*(U_x(x^*,y^*) - \mu^* p_x) = 0 \\ y^*\hat{\mathcal{L}}_y(x^*,y^*,\mu^*) &= y^*\big(U_y(x^*,y^*) - \mu^* p_y\big) = 0 \\ \hat{\mathcal{L}}_\mu(x^*,y^*,\mu^*) &= -\big(p_x x^* + p_y y^* - I\big) \ge 0 \\ \mu^*\hat{\mathcal{L}}_\mu(x^*,y^*,\mu^*) &= -\mu^*\big(p_x x^* + p_y y^* - I\big) = 0 \\ x^* &\ge 0 \\ y^* &\ge 0 \\ \mu^* &\ge 0. \end{aligned}$$

- If $p_x x^* + p_y y^* - I = 0$, the constraint is binding and $x^* > 0$ and $y^* > 0$.
- If $p_x x^* + p_y y^* - I < 0$, then $\mu^* = 0$. This implies

$$\begin{aligned} U_x(x^*,y^*) &\le 0 \\ U_y(x^*,y^*) &\le 0 \end{aligned}$$

  But this is unlikely to happen since usually the marginal utility is positive.
- If all money is spent and $x^* = 0$ and $y^* = 0$, then we have a corner solution. We have

$$\hat{\mathcal{L}}_x(x^*,y^*,\mu^*) = U_x(x^*,y^*) - \mu^* p_x < 0 \Longrightarrow U_x(x^*,y^*) < \mu^* p_x$$

and

$$\begin{aligned} &y^*\hat{\mathcal{L}}_y(x^*,y^*,\mu^*) = y^*\big(U_y(x^*,y^*) - \mu^* p_y\big) = 0 \\ &\Longrightarrow U_y(x^*,y^*) - \mu^* p_y = 0 \\ &\Longrightarrow U_y(x^*,y^*) = \mu^* p_y \end{aligned}$$

We thus have

$$\frac{U_x(x^*,y^*)}{U_y(x^*,y^*)} < \frac{p_x}{p_y}.$$

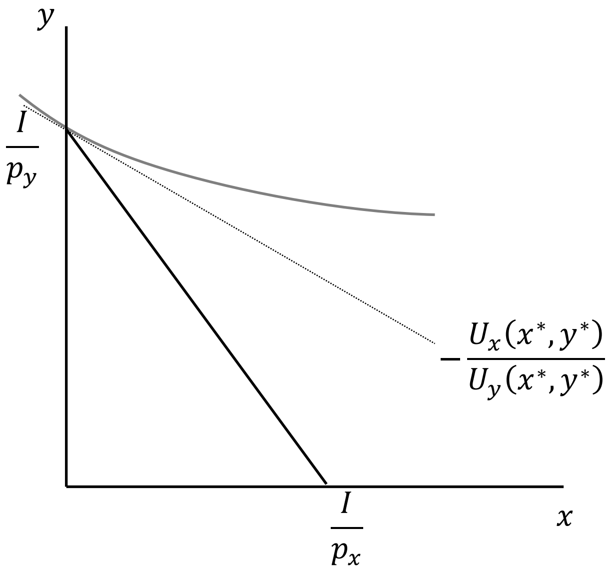


Figure 11.1 For corner solution, the slope of the budget line is not equal to the slope of indifferent curve.

HW. Write the K-T condition for the problem

$$\begin{aligned} &\max Z = U(\mathbf{x}) \\ &\text{st. } \mathbf{p}^{\mathrm{T}}\mathbf{x} \le I \\ &\quad \mathbf{t}^{\mathrm{T}}\mathbf{x} = T \\ &\quad \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

11.2 Two Goods Diet Problem

	% of Requirement	
Nutrients	Milk	Cereals
Vitamin A	6%	30%
Vitamin D	25%	25%
Calcium	30%	15%
Iron	0%	45%

Define the decision variables:

M = the number of glasses of milk
C = the number of bowls of cereals

The prices of a glass of milk and a bowl of cereals are 12 and 24 cents respectively. The linear programming problem is formulated as

$$\min Z = 12M + 24C$$

$$\begin{aligned} \text{st.}\, 6M + 30C &\geq 100 \quad (1) \\ 25M + 25C &\geq 100 \quad (2) \\ 30M + 15C &\geq 100 \quad (3) \\ 45C &\geq 100 \quad (4) \\ M, C &\geq 0. \end{aligned}$$

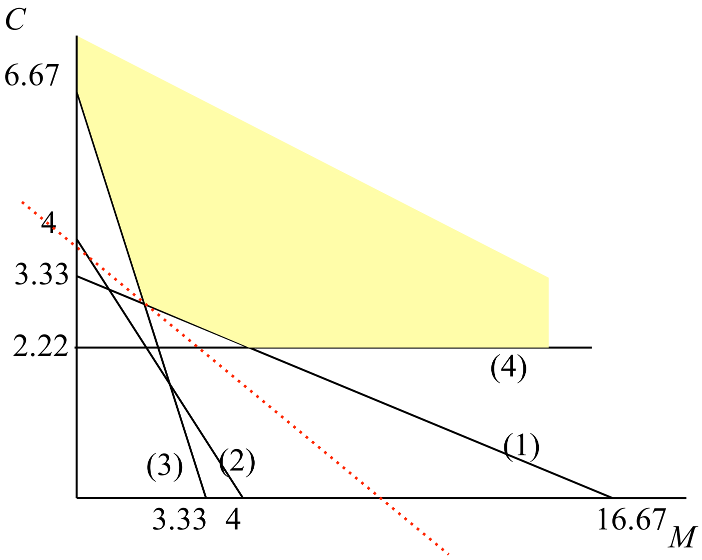
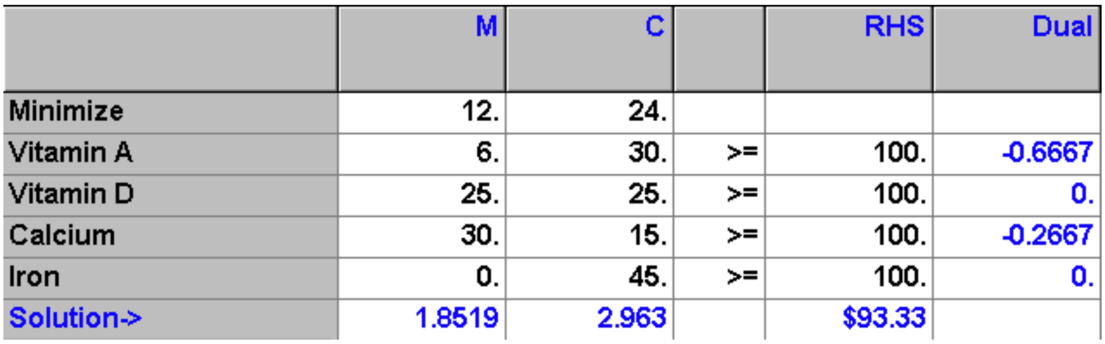


Figure 11.2 Graph of Two Goods Diet LP problem

- The minimum point is at the intersection of the first and third constraints at $M^* = \frac{50}{27}$ and $C^* = \frac{80}{27}$, with $Z = \frac{280}{3}$. This means the Lagrange multipliers associated with these two constraints are positive and the rest are zero.



	M	C		RHS	Dual
Minimize	12.	24.			
Vitamin A	6.	30.	>=	100.	-0.6667
Vitamin D	25.	25.	>=	100.	0.
Calcium	30.	15.	>=	100.	-0.2667
Iron	0.	45.	>=	100.	0.
Solution->	1.8519	2.963		$93.33	

Figure 11.3 Optimal Solution of Diet Problem

$$\begin{aligned} &\hat{\mathcal{L}}(M, C, \mu_1, \mu_2, \mu_3, \mu_4) \\ &\qquad = 12M + 24C - \mu_1(6M + 30C - 100) - \mu_2(25M + 25C - 100) \\ &\qquad - \mu_3(30M + 15C - 100) - \mu_4(45C - 100). \end{aligned}$$

K-T Condition

$$\begin{aligned} \hat{\mathcal{L}}_M(M^*, C^*, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*) &= 12 - 6\mu_1^* - 25\mu_2^* - 30\mu_3^* \geq 0 \\ \hat{\mathcal{L}}_C(M^*, C^*, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*) &= 24 - 30\mu_1^* - 25\mu_2^* - 15\mu_3^* - 45\mu_4^* \geq 0 \\ M^*\hat{\mathcal{L}}_M(M^*, C^*, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*) &= M^*(12 - 6\mu_1^* - 25\mu_2^* - 30\mu_3^*) = 0 \end{aligned}$$

$$
\begin{aligned}
C^*\hat{\mathcal{L}}_C(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= C^*(24-30\mu_1^*-25\mu_2^*-15\mu_3^*-45\mu_4^*)=0\\
\hat{\mathcal{L}}_{\mu_1}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= -(6M^*+30C^*-100)\le 0\\
\hat{\mathcal{L}}_{\mu_2}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= -(25M^*+25C^*-100)\le 0\\
\hat{\mathcal{L}}_{\mu_3}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= -(30M^*+15C^*-10\le 0\\
\hat{\mathcal{L}}_{\mu_4}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= -(45C^*-100)\le 0\\
\mu_1^*\hat{\mathcal{L}}_{\mu_1}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= \mu_1^*(6M^*+30C^*-100)=0\\
\mu_2^*\hat{\mathcal{L}}_{\mu_2}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= \mu_2^*(25M^*+25C^*-100)=0\\
\mu_3^*\hat{\mathcal{L}}_{\mu_3}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= \mu_3^*(30M^*+15C^*-10=0\\
\mu_4^*\hat{\mathcal{L}}_{\mu_4}(M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^*) &= \mu_4^*(45C^*-100)=0\\
M^*,C^*,\mu_1^*,\mu_2^*,\mu_3^*,\mu_4^* &\ge 0
\end{aligned}
$$

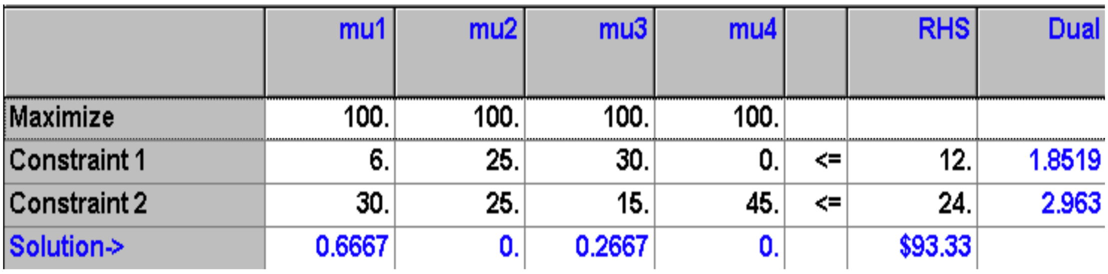
Since $M^*, C^* > 0$ and $\mu_2^*, \mu_4^* = 0$, we have

$$
\left.\begin{aligned} 12-6\mu_1^*-30\mu_3^* &= 0\\ 24-30\mu_1^*-15\mu_3^* &= 0\end{aligned}\right\} \Rightarrow \begin{cases} \mu_1^* = \dfrac{2}{3}\\ \mu_3^* = \dfrac{4}{15}\end{cases}
$$

Every linear programming problem has an associated problem called its dual problem. The original problem is called primal. If the primal is a minimization problem, its dual is a maximization one. The dual problem of this primal is given by

$$
\begin{aligned}
\max V &= 100\mu_1+100\mu_2+100\mu_3+100\mu_4\\
6\mu_1+25\mu_2+30\mu_3+0\mu_4 &\le 12 \quad (1)\\
30\mu_1+25\mu_2+15\mu_3+45\mu_4 &\le 24 \quad (2)\\
\mu_1,\mu_2,\mu_3,\mu_4 &\ge 0.
\end{aligned}
$$

Solving the dual problem yields the following result. Note the relationship between this dual problem and the primal one.



	mu1	mu2	mu3	mu4		RHS	Dual
Maximize	100.	100.	100.	100.			
Constraint 1	6.	25.	30.	0.	<=	12.	1.8519
Constraint 2	30.	25.	15.	45.	<=	24.	2.963
Solution->	0.6667	0.	0.2667	0.		$93.33	

Figure 11.4 Optimal Solution of Maximization of the Dual Problem

HW Baldani, p. 313, #12.3. Formulate the problem, write K-T conditions, and formulate its dual LP problem. Do not solve.

HW Baldani, p. 314, #12.5 Formulate the problem and write K-T conditions. Do not solve.

11.3 Sales Maximization When there is a separation of management and ownership, the management might aim to maximize sales with the condition that the profit is at least at an acceptable level. The problem is given by

$$\begin{aligned} &\max R(q) = R(f(L,K)) \\ &\text{st.}\, R(f(L,K)) - wL - rK \geq \pi_0 \\ &\qquad\qquad\qquad\qquad L, K \geq 0. \end{aligned}$$

The K-T Lagrange function is given by

$$\hat{\mathcal{L}}(L,K,\mu) = R\big(f(L,K)\big) - \mu(\pi_0 - R(f(L,K)) - wL - rK)$$

and the K-T conditions are given by

$$\begin{gathered} \nabla_{\left[\begin{smallmatrix} L \\ K \end{smallmatrix}\right]}\hat{\mathcal{L}}(L,K,\mu) = \begin{bmatrix} R'f_L(L^*,K^*) - \mu^* R' f_L(L^*,K^*) - \mu^* w \\ R'f_K(L^*,K^*) - \mu^* R' f_K(L^*,K^*) - \mu^* r \end{bmatrix} \leq \mathbf{0} \\ L^*\hat{\mathcal{L}}_L(L,K,\mu) = L^*(R'f_L(L^*,K^*) - \mu^* R' f_L(L^*,K^*) - \mu^* w) = 0 \\ L^*\hat{\mathcal{L}}_L(L,K,\mu) = K^*(R'f_K(L^*,K^*) - \mu^* R' f_K(L^*,K^*) - \mu^* r) = 0 \\ \hat{\mathcal{L}}_\lambda(L,K,\mu) = -\big(\pi_0 - R\big(f(L^*,K^*)\big) - wL^* - rK^*\big) \geq 0 \\ \mu^*\hat{\mathcal{L}}_\lambda(L,K,\mu) = \mu^*\big(\pi_0 - R\big(f(L^*,K^*)\big) - wL^* - rK^*\big) = 0 \\ L^*, K^*, \mu^* \geq 0 \end{gathered}$$

Assuming the optimal solution is where $L^*, K^* \geq 0$, we have

$$\left.\begin{aligned} \frac{1+\mu^*}{\mu^*} R' f_L(L^*,K^*) = w \\ \frac{1+\mu^*}{\mu^*} R' f_K(L^*,K^*) = r \end{aligned}\right\} \Rightarrow \left\{ \frac{f_L(L^*,K^*)}{f_K(L^*,K^*)} = \frac{w}{r}. \right.$$

If the constraint is binding, we will have $\mu^* > 0$, and then

$$\begin{aligned} \frac{1+\mu^*}{\mu^*} R' f_L(L^*,K^*) = w \Rightarrow MRP_L < w. \\ \frac{1+\mu^*}{\mu^*} R' f_K(L^*,K^*) = r \Rightarrow MRP_K < r. \end{aligned}$$

The last equation states that at the point where the sales is maximized, the labor is overemployed. The marginal revenue product of labor is less than its wage.

11.4 Intertemporal Consumption with Liquidity Constraint The consumer is not allowed to borrow in the first period in a 2-period model. The utility is maximized with the following inequality constraints.

$$\begin{aligned}
&\max U(c_1, c_2)\\
&\text{st.}\, c_1 \le I_1\\
&\quad c_2 \le I_2 + (I_1 - c_1)(1+r)\\
&c_1, c_2 \ge 0.
\end{aligned}$$

The second constraint can be written as

$$(1+r)c_1 + c_2 \le I_2 + I_1(1+r)$$

and the K-T Lagrange function is given by

$$\hat{\mathcal{L}}(c_1, c_2, \mu_1, \mu_2) = U(c_1, c_2) - \mu_1(c_1 - I_1) - \mu_2\big((1+r)c_1 + c_2 - I_2 - I_1(1+r)\big).$$

K-T conditions are

$$\begin{aligned}
\nabla_{\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]} \hat{\mathcal{L}}(c_1^*, c_2^*, \mu_1^*, \mu_2^*) &= \begin{bmatrix} U_1(c_1^*, c_2^*) - \mu_1^* - \mu_2^*(1+r) \\ U_2(c_1^*, c_2^*) - \mu_2^* \end{bmatrix} \le \mathbf{0}\\
c_1^*\big(U_1(c_1^*, c_2^*) - \mu_1^* - \mu_2^*(1+r)\big) &= 0\\
c_2^*(U_2(c_1^*, c_2^*) - \mu_2^*) &= 0\\
\nabla_{\left[\begin{smallmatrix} \mu_1 \\ \mu_2 \end{smallmatrix}\right]} \hat{\mathcal{L}}(c_1^*, c_2^*, \mu_1^*, \mu_2^*) &= \begin{bmatrix} -(c_1^* - I_1) \\ -\big((1+r)c_1^* + c_2^* - I_2 - I_1(1+r)\big) \end{bmatrix} \ge \mathbf{0}\\
-\mu_1^*(c_1^* - I_1) &= 0\\
\mu_2^*\big((1+r)c_1^* + c_2^* - I_2 - I_1(1+r)\big) &= 0\\
c_1^*, c_2^*, \mu_1^*, \mu_2^* &\ge 0
\end{aligned}$$

If $U(c_1, c_2) = c_1 c_2$ and assuming that $0 < c_1^* < I_1$ and $c_2^* > 0$, we have $\mu_1^* = 0$ and

$$\left.\begin{array}{c} c_1^* - \mu_1^* - \mu_2^*(1+r) = 0 \\ c_1^* - \mu_2^* = 0 \\ \big((1+r)c_1^* + c_2^* - I_2 - I_1(1+r)\big) = 0 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{c} c_1^* = \mu_2^* \\ c_2^* - c_1^*(1+r) = \mu_1^* = 0 \end{array}\right.$$

HW If the optimal solution is found at $c_1^* = I_1$ and $c_2^* > 0$, show that $\frac{U_1(c_1^*, c_2^*)}{U_2(c_1^*, c_2^*)} > 1 + r$, for a general form of utility function $U(c_1, c_2)$.