EE 320 Introductory Mathematical Economics

Semester 1/2017

Assignment# 5

Solution

Question 1 Output Maximization

A firm's production function is given by

$$Q = 2x^2 - 2xy + y^2$$

where x and y are the two inputs, and the per unit prices for x and y are \$10 and \$20, respectively. Suppose that the firm's total cost of this production is fixed at \$5200.

a. Show that the isoquant derived from the above production function is convex.

Q is convex if d^2Q is positive definite. (d^2Q is positive definite if and only if $Q_{xx} > 0;\ Q_{yy} > 0;\ and\ Q_{xx}Q_{yy} - (Q_{xy})^2 > 0$)

$Q_{xx} = 4;\ Q_{yy} = 2$; and $Q_{xx}Q_{yy} - (Q_{xy})^2 = 4$.

Hence, Q is convex.

b. Based on the Lagrangian function, write down the first-order conditions for this output maximization problem, and explain the first-order conditions in terms of isoquant and isocost.

$$L = 2x^2 - 2xy + y^2 + \lambda[5200 - 10x - 20y]$$

FONC:

$L_x = 4x - 2y - 10\lambda = 0$ -- (1)

$L_y = -2x + 2y - 20\lambda = 0$ -- (2)

$L_\lambda = 5200 - 10x - 20y = 0$ --(3)

(1) & (2) $\Rightarrow \frac{MP_L}{MP_K} = \frac{w}{r}$ (ie. slope of isoquant = slope of isocost).

c. From the results in part (b), find the input levels that maximize output, and determine the maximum output level as well as the Lagrange multiplier.

$x^* = 120$; $y^* = 200$; $\lambda^* = 8$

$Q^* = 20{,}800$

d. Suppose now that the firm's total cost increases to 5201 baht. What happens to the optimal production levels? How does this change in the optimal production levels relate to the Lagrange multiplier in part (c)?

$\lambda^* = \frac{dQ^*}{dc} = 8 \Rightarrow \Delta Q^* \approx \lambda^* \Delta c = 8$. Hence, new Q* = 20,808.

[Note: you can double check your answer by plugging in 5201 in equation (3) in FONC.]

<u>Question 2</u> Constrained Optimization with 4 variables and 2 constraints

Consider a constrained optimization problem with the objective function

$$z = f(x_1, x_2, x_3, x_4)$$

Subject to the two constraints:

$$g(x_1, x_2, x_3, x_4) = c$$
$$h(x_1, x_2, x_3, x_4) = d$$

a. Write down the Lagrangian function and the corresponding first-order conditions.

$$L(x_1,x_2,x_3,x_4) = f(x_1,x_2,x_3,x_4) + \lambda[c - g(x_1,x_2,x_3,x_4)] + \mu[d - h(x_1,x_2,x_3,x_4)]$$

FONC:

$$L_{x_i} = \frac{\partial f}{\partial x_i} - \lambda\frac{\partial g}{\partial x_i} - \mu\frac{\partial h}{\partial x_i} = 0$$

$$L_\lambda = c - g(x_1,x_2,x_3,x_4) = 0$$

$$L_\mu = d - h(x_1,x_2,x_3,x_4) = 0$$

b. Write down the bordered Hessian matrix for this constrained optimization problem.

$$\bar{H} = \begin{bmatrix} 0 & 0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & h_1 & h_2 & h_3 & h_4 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ g_4 & h_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix}$$

c. Using the notations from the lecture, indicate the second-order conditions for a maximum <u>and</u> a minimum of z.

Max: $|\bar{H}_3| < 0$ & $|\bar{H}_4| = |\bar{H}| > 0$

Min: $|\bar{H}_3| > 0$ & $|\bar{H}_4| = |\bar{H}| > 0$

Where

$$\bar{H}_3 = \begin{bmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{bmatrix}$$

Question 3 Integration

Suppose the demand curve for a product is given by $Q_d = 20 - 4P$ and the supply curve is given by $Q_s = -4 + 2P$.

a. Use integral to calculate the consumer and producer surplus at the equilibrium price and quantity.

At equilibrium, Qd = Qs. ⇒ P*=4 & Q*=4

Derive inverse demand and supply functions:

$P_d = 5 - 0.25Q_d$

$P_s = 2 + 0.5Q_s$

$$CS = \int_0^4 [(5 - 0.25Q) - 4]dQ = 4 - 0.125(4)^2 = 2$$

$$PS = \int_0^4 [4 - (2 + 0.5Q)]dQ = 2(4) - 0.25(4)^2 = 4$$

Total welfare = CS+PS = 6.

b. Suppose the government imposes a *25 %* ad varolem tax on the consumer. Derive the new demand curve, and use integral to calculate the consumer and producer surplus at the new equilibrium price and quantity. Discuss the changes in both consumer and producer surplus. Is there any deadweight loss to the society?

$P^T = (1 + 0.25)P$ where $P = P_s$

New equilibrium: $20 - 4[(1 + 0.25)P] = -4 + 2P$

⇒ $P^*_s = 3.43$ and $P^*_d = 4.29$

⇒ $Q^*_T = 2.86$

New inverse demand: $P = 4 - 0.2Q'_d$

$$CS^T = \int_0^{2.86} [(5 - 0.25Q) - 4.29]dQ = 1.008$$

$$PS^T = \int_0^{2.86} [3.43 - (2 + 0.5Q)]dQ = 2.04$$

$$\Delta CS \approx -1$$

$$\Delta PS \approx -1.96$$

<u>Question 4</u> Cost function under the multi-plant problem.

A company has *three* plants that it can use for producing a product. Given the level of output chosen in each of the three plants, the operating cost incurred under each plant can be given,

$$c_1(x) = 200 + \tfrac{1}{100}x^2\ ,\ c_2(y) = 200 + y + \tfrac{1}{300}y^2\ ,\ c_3(z) = 200 + 10z\ ,$$

where x, y and z are the amount of output chosen in each plant, respectively.

a) Suppose that the total level of production required is equal to Q units. Calculate the optimal plant size that minimizes the total cost. Determine the minimized level of cost.

$$L = 200 + \tfrac{1}{100}x^2 + 200 + y + \tfrac{1}{300}y^2 + 200 + 10z + \lambda(Q - x - y - z)$$

FOC: $L_x = L_y = L_z = L_\lambda = 0$

(1) $L_x = x/50 - \lambda = 0$

(2) $L_y = 1 + y/150 - \lambda = 0$

(3) $L_z = 10 - \lambda = 0$

(4) $L_\lambda = Q - x - y - z = 0$

Note from (3) that $\lambda = 10$. Therefore, we yield that x = 500 and y = 1350. Using (4), we obtain that z = Q - 1850.

b) Confirm your result with the second-order derivative test when Q = 2000.

Note first that z = 150 when Q = 2000.

H = [0 -1 -1 -1;
-1 1/50 0 0;
-1 0 1/150 0;
-1 0 0 0]

#constraint = 1; #variable =3
The minimum bordered hessian matrix is 2*#constraint +1 = 3.
So, we must check H3 and H4.

To warrant the minimization, both matrixes must produce the same sign of determinant, where the sign must be (-1)^#constraint < 0. That is, determinant of H3 and H4 must be negative both.

c) How does the minimized level of cost respond to the change in total level of production (Q)?
Note that $\lambda = 10$. By the envelop theorem, we know that $\frac{\partial c^*}{\partial Q} = \lambda = 10$.

Question 5

Consider a utility maximization problem where household chooses for the optimal combination of good 1 (x) and good 2 (y). Suppose that p_x and p_y are the prices per unit of good 1 and good 2, respectively. Assume that the household has the budget equal to M, and the utility function is given by $U(x,y) = \sqrt{x} + y$. Consider the following problem

a) Derive for the bundle of consumption that maximizes household's utility, i.e. demand for good 1 and demand for good 2.

$$L = \sqrt{x} + y + \lambda(M - P_x x - P_y y)$$

The first-order conditions are:

(1) $L_x = 1/2\sqrt{x} - P_x\lambda = 0$

(2) $L_y = 1 - P_y\lambda = 0$

(3) $L_\lambda = M - P_x x - P_y y = 0$

Note from (2) that $\lambda = 1/P_y$.

This implies that, from (1), we yield $x = 0.25 * (P_y/P_x)^2$.

By using (3), we obtain that

$y = [M - P_x 0.25 * (P_y/P_x)^2]/P_y = (M/P_y) - 0.25 * (P_y/P_x)$

b) State the condition under which both types of good are chosen.

Note first that x is always chosen.

Y is set to zero if $(M/P_y) - 0.25 * (P_y/P_x)$ is less than or equal to zero.

c) Derive the maximized level of utility.

Substituting x and y that were derived above into the utility function, we yield the indirect utility function, the maximized level of utility,

$$v(p_x, p_y, M) = 0.5 * (P_y/P_x) + (M/P_y) - 0.25 * (P_y/P_x)$$

Question 6

Suppose that a monopolist faces with a marginal revenue function given by

$$MR(q) = 25 - 2q,$$

and the marginal cost function given by,

$$MC(q) = 37 - 9q + q^2,$$

where q is the unit of output. Assume that fixed cost is $7. Consider the following problem.

a) Derive the revenue function and infer the functional form of the market demand curve.

$R(q) = 25q - q^2$ (note we make use the fact that R=0 when q=0)

$P = 25 - q$

b) Derive the total cost function.

$C(q) = 7 + 37q - 9q^2/2 + q^3/3$ (note that we make use the fact that C=7 when q=0)

c) Determine the level of production that maximizes profit and determine the level of maximized profit.

$MR = MC \rightarrow q = 3, 4$

Note that q = 4 produces negative second-derivative of the profit function.

At the quantity, price is equal to 21

Total revenue is $84. Profit is then equal to 84 - 104.3= - $20.3

Notice carefully that if firm chooses to produce q = 4, firm would have to incur more loss than not producing. Based on the calculation, TVC is $97.3 and TC is $104.3. As a result, shutting-down plant would incur loss by -$7, which is less than

-\$20.3. Hence, it is in fact optimal for firm to shut-down, and choose q =0. Profit-maximizing level of output should in fact be q=0.

d) Calculate the consumer surplus and producer under the monopoly equilibrium.

Correct answer should be that CS = 0 and PS = 0 as the equilibrium is q = 0.

But, let's pretend to be a naive guy that doesn't care about the logic reasoning described in the above part, and calculate the surplus using the standard integration technique that we learned in class. (This is for the purpose that I show you how to do it. But, keep in mind that the correct answer is that both CS and PS are zero!!)

CS = area under demand - total spending =

$$\int_0^4 (25 - q)dq - 21 * 4 = [25q - q^2/2]_0^4 - 84 = 100 - 8 - 84 = 8$$

PS = total spending - area under marginal cost

$$21 * 4 - \int_0^4 (37 - 9q + q^2)dq = 84 - (37(4) - 9/2 * (4)^2 + (1/3)(4)^3) = -13.3$$

e) If the monopolist were to act as a perfectly competitive firm, calculate the market equilibrium, and corresponding level of consumer's and producer's surplus.

If the monopolist were to act as a perfectly competitive firm, the monopolist would set "q" where P is equal to MC. That is, q = 2 or 6. If one checks the second-order condition, we will find that q=6 satisfies the negative second-order derivative of profit function. At this q =6, price would be $19. Hence, total revenue is $114. Total cost is equal to $139. As a result profit would be negative $25, i.e. losing money by $25.

Again, this would run into the similar problem to part c where you incur loss, and the loss is higher than if you shut-down your plant. Hence, the correct answer is that q = 0 as well. Thus, both CS and PS would be zero, again!

But, let's once again pretend that we don't care about the shutdown problem, and go on deriving the CS and PS as if q =6 were the correct equilibrium.

CS = area under demand - total spending =

$$\int_0^6 (25-q)dq \; - 19*6 = 25q - q^2/2]_0^6 - 114 = 150 - 18 - 114 \; = 18.$$

PS = total spending - area under marginal cost

$$114 - \int_0^6 (37 \; - 9q \; + \; q^2)dq \; = 114 - (37(6) - 9/2*(6)^2 + (1/3)(6)^3) = \; 114 - 151 =- 37$$